Signals and Systems

Module-1

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Introduction and Classification of Signals

Introduction:-

Signals, in one form (or) another, constitute a basic ingredient of our daily lives. For example, a common form of human communication takes place through the use of speech signals, in a face-to-face conversation (or) over a telephone channel.

Another common form of human communication is visual in nature with the signals taking the form of images of people (or) objects around us.

Another form of human communication is through electronic mail over the internet. In addition to mail, the internet provides a powerful medium for searching for information of general interest, advertising, telecommuting, education and games. All of these forms of communication over the internet involve the use of information-bearing signals of one kind (or) another.

Signal :-

A signal is defined as any physical quantity that varies with time, space, frequency (or) any other independent variable.

(or)

Signal is defined as a function of one (or) more independent variables which conveys a certain information.

Signals are represented mathematically as a function of one (or) more independent variables. i.e., Signal $= f(x_1, x_2, x_3, \ldots)$.

If the function depends on one independent variable, then the signal is said to be one-dimensional signal.

$\boxed{f(t) = 7t}$ Ex :- Speech signal, whose amplitude varies with time, depending on the spoken word and a person who speaks it.

If a function depends on two-independent variables, then the signal is known as two-dimensional signal.

$\boxed{f(x,y) = 6x + 7y + 10xy}$ Ex: Image, with the horizontal and vertical co-ordinates of the image representing the two dimensions.

Some of the signals which cannot be expressed by simple mathematical equations. Such as, AC power supply signal, Speech signal, Electrocardiogram, the variation in temperature at a point in a furnace and sound of an automobile (one-dimensional signals).

If a function depends on two (or) more independent variables, then the signal is said to be multi-dimensional signal.

$\boxed{f(x,y,z),\ f(x,y,z,t)}$ Ex:- 3D Image, Video.

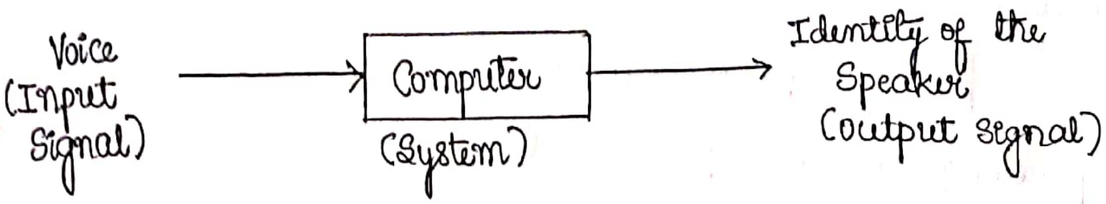
Systems :-

A system is defined as an entity that manipulates one (or) more signals to accomplish a function, thereby yielding a new signals.

(or)

System is a physical device that performs an operation on the information-bearing signal.

In an automatic speaker recognition system, the input signal is a speech (voice) signal, the system is a computer, and the output signal is the identity of the speaker.



Classification of Signals :-

Based on different feature, the signals are classified as:

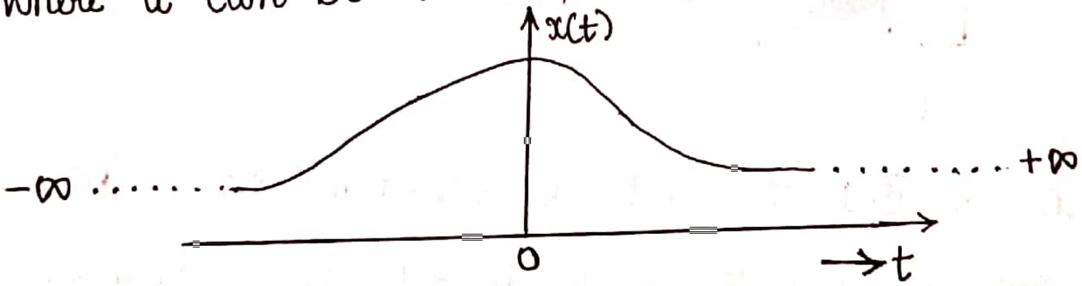
1. Continuous-time signal (CTS) and discrete-time signal (DTS).
2. Even and odd signal.
3. periodic signal and Non-periodic signal.
4. Deterministic signal and Random signal.
5. Energy signal and power signal.

1) Continuous-time Signal and Discrete-time Signal :-

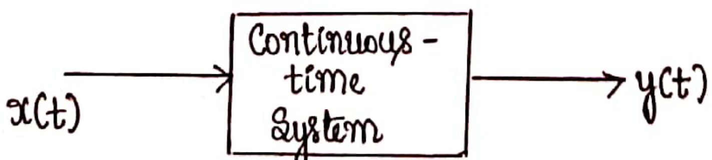
A signal $x(t)$ is said to be a continuous-time signal if it has value of amplitude for all time 't' (i.e., the independent variable 't' is continuous).

(or)

Continuous-time signals are defined for every value of time and they take on values in a continuous interval (a, b), where 'a' can be $-\infty$ and 'b' can be $+\infty$.



where, 't' is independent variable and '$x(t)$' is dependent variable.



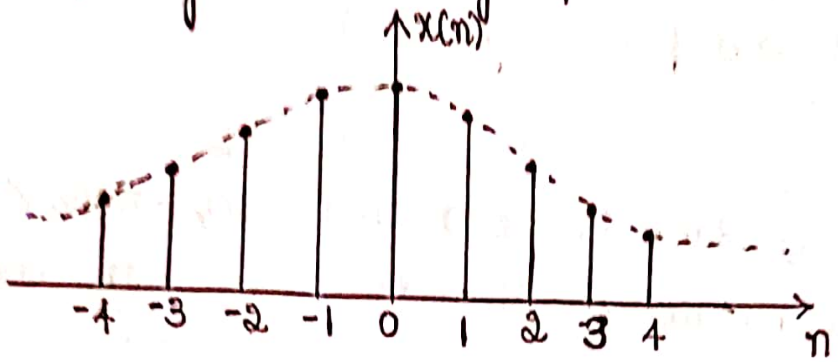
Note :-

1) usually $x(t)$ is used to denote continuous-time signal.
2) 't' is used to denote continuous-time variable.

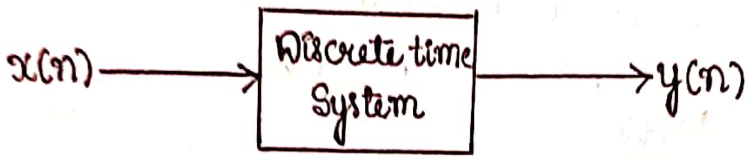
Discrete-time signal is defined only at discrete-instants of time (i.e., the independent variable has discrete-values only which are usually uniformly spaced).

(or)

Discrete-time signals are defined only at certain specific values of time. These time intervals need not be equidistant but in practice, they are usually spaced intervals for computation.



Where 'n' is independent variable and x(n) is dependent variable.



Note :-

1) usually, x(n) is used to denote discrete-time signal (DTS).

2) 'n' is an integer, used to denote discrete-time variable.

3) Discrete-time signal is obtained by sampling continuous-time signal at uniform rate.

Note : Representation of DTS :

We can represent DTS into

1. Graphical representation method.
2. Tabular representation method.
3. Sequential representation method.
4. Functional representation method.

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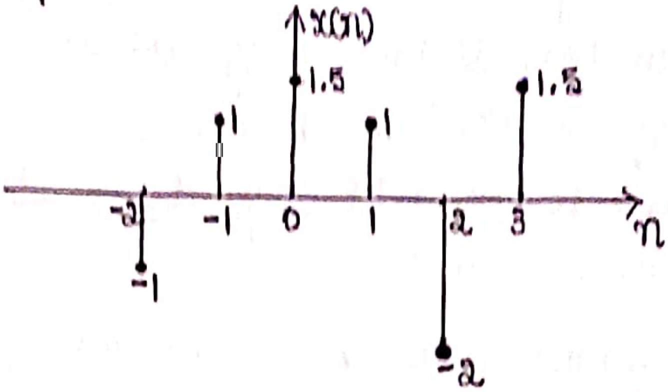
1. Graphical representation method :- $x(n) = \{-1, 1, \underset{\uparrow}{1.5}, 1, -2, 1.5\}$



2. Tabular representation method :

Above signal is Tabular representation is as follows :

n	-2	-1	0	1	2	3
$x(n)$	-1	1	1.5	1	-2	1.5

$n \to$ integer (no. of sample)

$x(n) \to$ amplitude of signal

3. Sequential representation method :

$$x(n) = \{\overset{x(-2)}{-1}, \overset{x(-1)}{1}, \underset{\uparrow}{\overset{x(0)}{1.5}}, \overset{x(1)}{1}, \overset{x(2)}{-2}, \overset{x(3)}{1.5}\}$$

Up-arrow indicates signal amplitude at $n=0$.

If up-arrow is not present, 1st sample itself is taken as $x(0)$.

i.e., $x(n) = \{5, 2, -1, 0\}$

then $x(0) = 5$; $x(1) = 2$; $x(2) = -1$; $x(3) = 0$.

4. Functional representation method :

$$\text{If } x(n) = \begin{cases} n+2 \ ; & \text{for } 0 \le n \le 2 \\ 0 \ ; & \text{else} \end{cases}$$

then $x(n) = \{\underset{\uparrow}{2}, 3, 4\}$ (or) $x(n) = \{2, 3, 4\}$

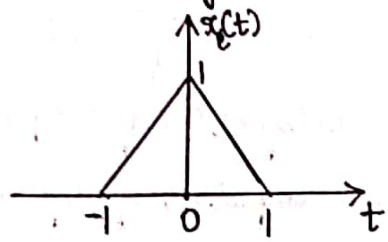
Even signal and odd signal :-

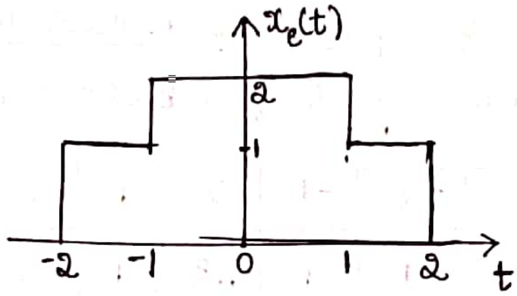
A continuous-time signal $x(t)$ is said to be an even signal if it satisfies the condition $x(-t) = x(t)$ for all 't' and it is said to be an odd signal if it satisfies the

Condition $x(-t) = -x(t)$ for all 't'.

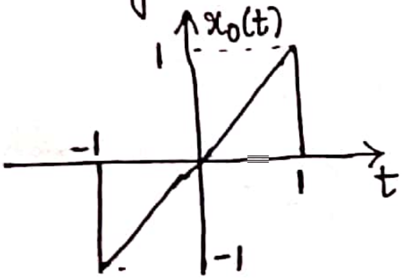
Similarly, a discrete-time signal $x(n)$ is said to be an even signal if it satisfies the condition $x(-n) = x(n)$ for all 'n' and it is said to be odd signal, if it satisfies the condition $x(-n) = -x(n)$ for all 'n'.

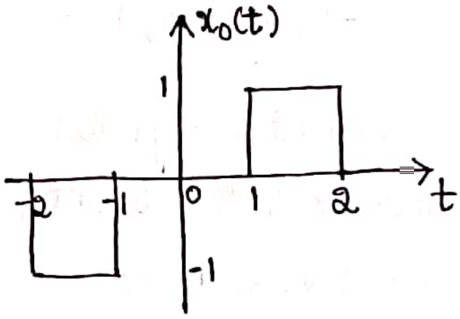
Even signals are symmetrical about the vertical axis (or) the time origin.





Odd signals are antisymmetric about the vertical axis (or) the time origin.





The above definitions of even and odd signal holds good for only real-valued signals. But complex-valued signals are characterized by conjugate symmetry.

A complex-valued signal $x(t)$ is said to be conjugate symmetric if it satisfies the condition,

$$x(-t) = x^*(t)$$

where, '$*$' denotes complex conjugate.

Let $x(t) = x_R(t) + j\,x_I(t)$

$x_R(t) \rightarrow$ Real part of $x(t)$ and $x_I(t) \rightarrow$ Imaginary part of $x(t)$

then $x^*(t) = x_R(t) - j\,x_I(t)$

Decomposition of a Signal :-

A Continuous-time signal $x(t)$ can be decomposed into a sum of two signals, one of which is Even $x_e(t)$ and the other is odd $x_o(t)$, Such that

$$x(t) = x_e(t) + x_o(t) \longrightarrow (1)$$

For $x_e(t)$ to be Even, $x_e(-t) = x_e(t)$

For $x_o(t)$ to be odd, $x_o(-t) = -x_o(t)$

put $t=-t$ in Eq(1),

$$x(-t) = x_e(-t) + x_o(-t)$$

$$\therefore x(-t) = x_e(t) - x_o(t) \longrightarrow (2)$$

Adding Eq(1) and Eq(2), we get

$$\left\langle x_e(t) = \frac{1}{2}\left[x(t) + x(-t)\right] \right\rangle$$

Subtracting Eq(1) from Eq(2), we get

$$\left\langle x_o(t) = \frac{1}{2}\left[x(t) - x(-t)\right] \right\rangle$$

Similarly, a discrete-time signal $x(n)$ can be decomposed into a sum of two signals, one of which is Even $x_e(n)$ and the other is odd $x_o(n)$, such that

$$x(n) = x_e(n) + x_o(n)$$

Where, $x_e(n) = \frac{1}{2}\left[x(n) + x(-n)\right]$

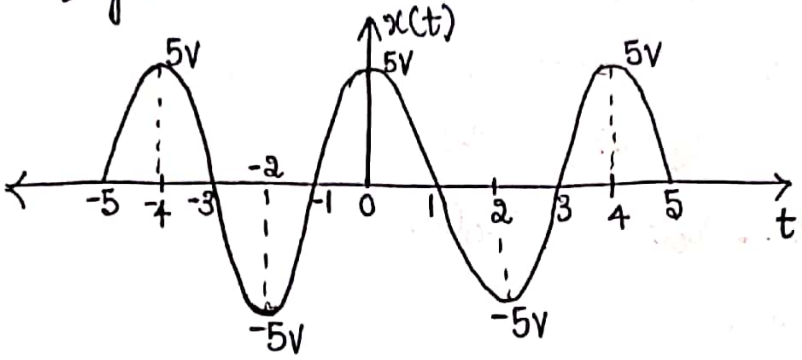
and $x_o(n) = \frac{1}{2}\left[x(n) - x(-n)\right]$

Note :-

1) Signal which has same amplitude both in positive and negative values of time is said to be Even signal.

Ex :- Cos signal

$$\boxed{x(t) = x(-t)}$$



$\therefore x(+2) = -5V$

$x(-2) = -5V$

and

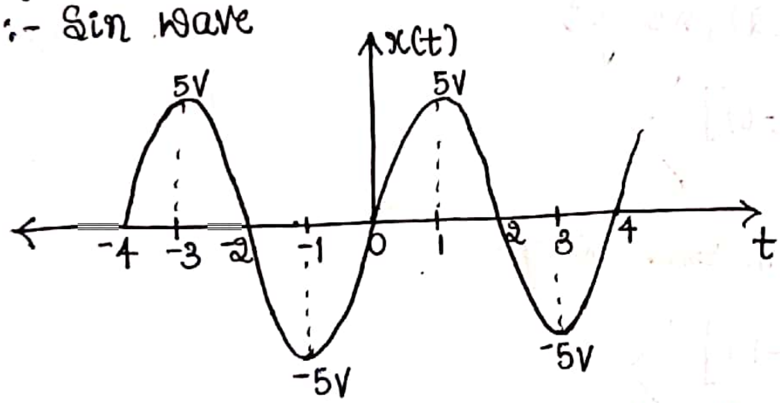
$x(4) = 5V$

$x(-4) = 5V$

2) Signal which has opposite amplitude both in positive and negative values of time is said to be odd signal.

Ex :- Sin wave

$$\boxed{x(t) = -x(-t)}$$



$x(1) = 5V$

$x(-1) = -5V$

2) Periodic Signal and Non-periodic Signal :-

A Continuous-time signal $x(t)$ is said to be periodic signal if it satisfies the condition, $x(t) = x(t+T)$ for all 't'. where, 'T' is a positive constant.

If the condition $x(t) = x(t+T)$ is satisfied for $T = T_0$, then it is also satisfied for any $T = nT_0$ where $n = 1, 2, 3, \ldots$.

The smallest value of 'T' that satisfies $x(t) = x(t+T)$ is called the fundamental period of $x(t)$.

(or)

The time taken by the signal $x(t)$ to complete its one cycle is called fundamental period, 'T'.

The reciprocal of the fundamental period 'T' is known as the fundamental frequency of the signal.

i.e., fundamental frequency $= \frac{1}{T}$ (Hz)

The fundamental angular frequency 'w' is given by,

$$w = 2\pi f = \frac{2\pi}{T} \text{ (rad/sec)}$$

Ex:

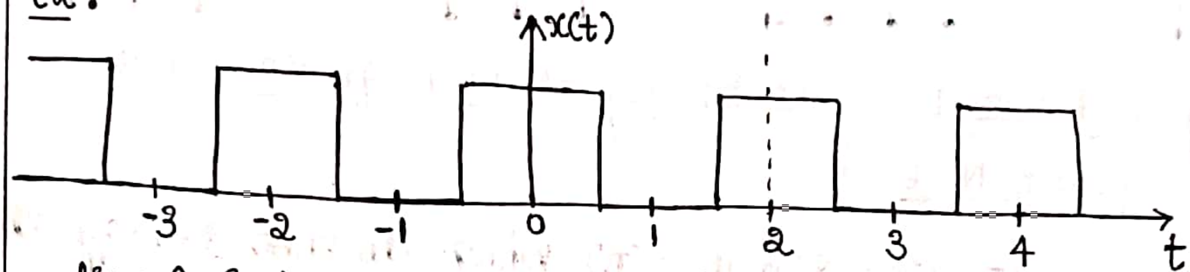


fig: A Continuous-time periodic signal with the fundamental period T = 2

Any continuous-time signal x(t) which does not satisfy $x(t) = x(t+T)$ is called non-periodic (aperiodic) signal.

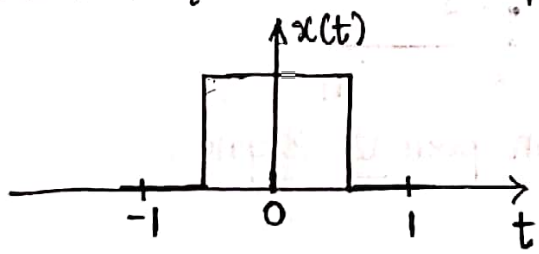


fig: Continuous-time non-periodic signal

Similarly, a discrete-time signal x(n) is said to be periodic signal if it satisfies the condition, $x(n) = x(n+N)$ for all 'n'.

where, 'N' is a positive integer.

The smallest value of 'N' which satisfies $x(n) = x(n+N)$ is called the fundamental period of the signal x(n).

The fundamental angular frequency of $x(n)$ is given by,

$$\Omega = \frac{2\pi}{N} \text{ (radians)}$$

Ex :-

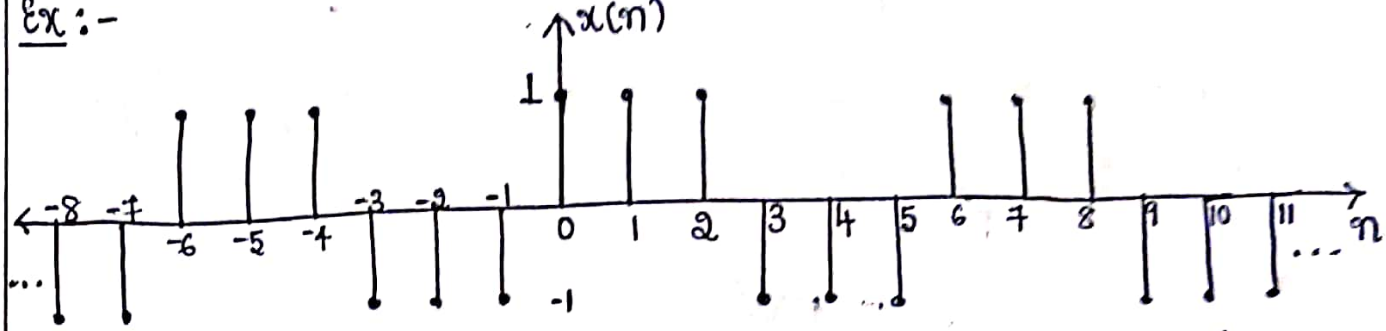


fig : A discrete-time periodic signal with fundamental period $N=6$

Any discrete-time signal $x(n)$ which doesnot satisfy the condition $x(n) = x(n+N)$ is called Non-periodic/aperiodic signal.

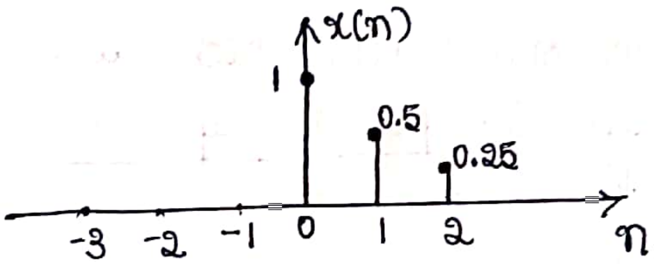


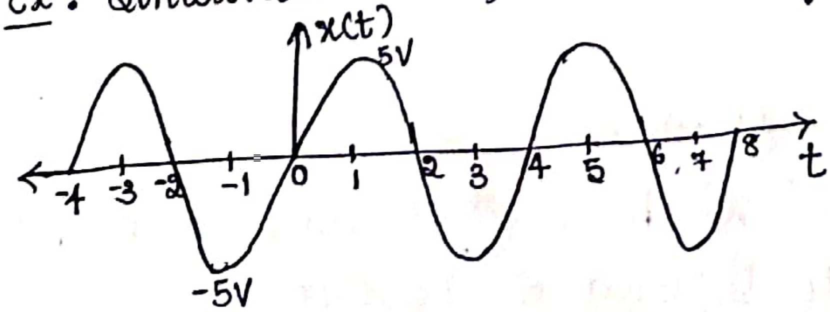
fig : A discrete-time non-periodic signal

Note :-

1) Signal which repeats its amplitude for time period is said to be periodic signal.

Ex : Sinusoidal wave, oscillation of pendulum.

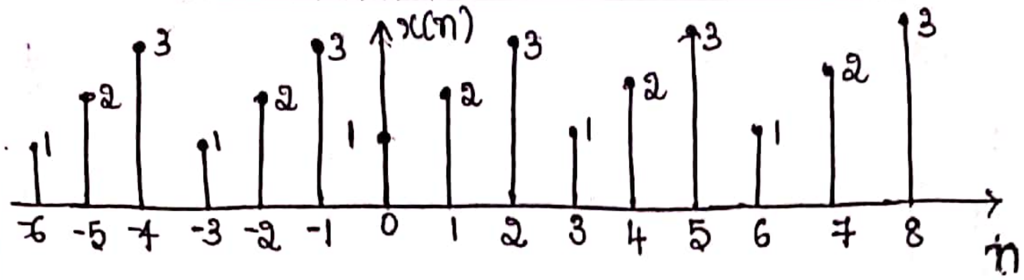
$$\boxed{x(t) = x(t+T)}$$



Time period $T=4$,

$x(1) = x(1+4) = x(5) = 5V$

$x(3) = x(3+4) = x(7) = -5V$



$\boxed{x(n) = x(n+N)}$

Time period $N = 3$,

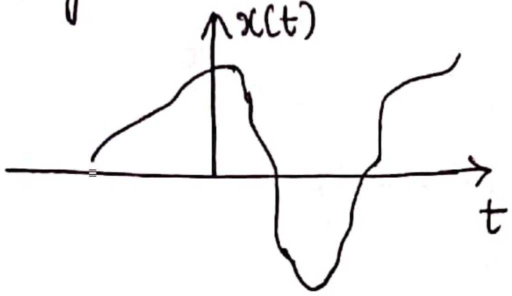
$x(0) = x(3) = 1$

$x(1) = x(4) = 2$

$x(2) = x(5) = 3$

2) Signal which doesnot repeates its amplitude is said to be <u>non-periodic signal</u>.

<u>Ex</u>: Audio signal.



$\boxed{x(t) \neq x(t+T)}$

<u>Time period for periodic signal</u> :

a) <u>Continuous time signal</u> :

Consider, Continuous periodic signal $x(t) = e^{j\omega t}$

For periodic signal, $x(t) = x(t+T)$

$$\therefore e^{j\omega t} = e^{j\omega(t+T)}$$

$$e^{j\omega t} = e^{j\omega t} \cdot e^{j\omega T}$$

$$e^{j\omega T} = 1 \longrightarrow (1)$$

$$e^{0} = 1$$

$$e^{j2\pi} = \cos 2\pi + j \sin 2\pi = 1$$

$$e^{j4\pi} = 1$$

$$e^{j6\pi} = 1$$

$$\therefore e^{jm2\pi} = 1 \longrightarrow (2) \text{ if } m = 0, 1, 2, 3, \ldots$$

periodic (CTS)	Non-periodic (DTS)
$T = \pi$	$T = 1/\pi$
$T = 2\pi$	$T = 1/2\pi$
$T = 1, 2, 3, \ldots$	$T = \sqrt{2}$
$\frac{T_1}{T_2} = 2.5$	$\frac{T_1}{T_2} = \frac{17}{31} = 0.548387\ldots$
$\frac{T_1}{T_2} = \frac{1}{3} = 0.3333$	$\frac{T_1}{T_2} = \frac{1}{7} = 0.1428571\ldots$

Comparing Eq(1) and Eq(2),

$$WT = m2\pi$$

$$\therefore \boxed{T = m \cdot \frac{2\pi}{W}}$$

If $m=1$,

$$\boxed{T = \frac{2\pi}{W}}$$ Fundamental time period (Minimum possible time period).

b) Discrete-time signal :-

Consider discrete periodic signal $x(n) = e^{j\Omega n}$

$W \rightarrow$ Continuous angular frequency

$\Omega \rightarrow$ Discrete angular frequency

$x(n) = x(n+N)$, for periodic DTS

$$\therefore e^{j\Omega n} = e^{j\Omega(n+N)}$$

$$e^{j\Omega n} = e^{j\Omega n} \cdot e^{j\Omega N}$$

$$e^{j\Omega N} = 1 \longrightarrow (1)$$

Periodic (DTS)	Non-periodic (DTS)
$N = 1, 2, 3, \ldots$	$N = \pi$
	$N = \sqrt{2}$
$\frac{m}{N} = 5/2, 3/8$	$\frac{m}{N} = 5/\sqrt{2}, \sqrt{3}/7$
Rational	Irrational
$5/2$ $3/8$	$5/\sqrt{2}$ $\sqrt{3}/7$

$$e^{0} = 1$$

$$e^{j2\pi} = \cos 2\pi + j \sin 2\pi = 1$$

$$e^{j4\pi} = 1$$

$$\therefore e^{jm2\pi} = 1 \longrightarrow (2) \quad \text{for } m = 0, 1, 2, 3, \ldots$$

Comparing Eq(1) and Eq(2),

$$\Omega N = m2\pi$$

$$\therefore \boxed{\frac{m}{N} = \frac{\Omega}{2\pi}}$$

Note: 'm' and 'N' must be integer to satisfy the periodic condition.

Note :

1) General format for Continuous :

* $x(t) = e^{j\omega t}$

* $x(t) = e^{j(\omega t + \theta)}$

* $x(t) = \cos \omega t$

* $x(t) = \sin(\omega t + \theta)$

* $x(t) = A\cos(\omega t + \phi)$

2) General format for Discrete:

* $x(n) = e^{j\Omega n}$

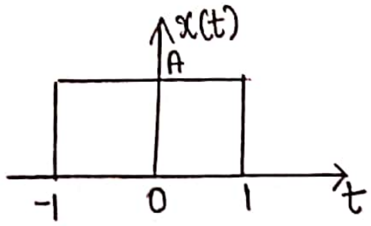
* $x(n) = e^{j(\Omega n + \phi)}$

* $x(n) = \cos(\Omega n)$

* $x(n) = A\sin(\Omega n + \theta)$

4) Deterministic Signals and Random Signals :-

A signal which can be uniquely described by the explicit mathe--matical representation, a table of data (or) a well-defined rule is called a deterministic signal.



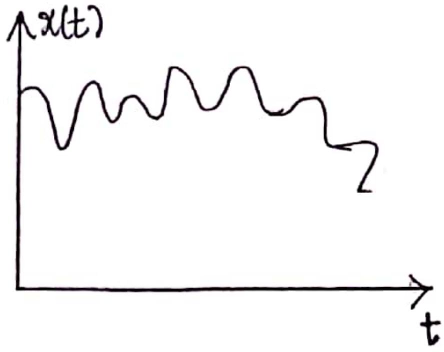
$$x(t) = \begin{cases} A; & -T<t<T \\ 0; & -\infty<t<T \\ & T<t<\infty \end{cases}$$

A deterministic signal behaves in a fixed known way with respect to time. It can be modelled as a function of time 't' (i.e., Continuous-time signal) (or) a function of a sample number 'n' (i.e., discrete-time signal).

To model deterministic signal mathematically, the range of values for 't' (or) 'n' must be specified. Otherwise, it valids for all values of 't' (or) 'n'.

Ex: Sine wave, $x(t) = \cos(\omega t)$ (CTS), $x(n) = \cos(\Omega n)$ (DTS).
Exponential pulse, triangular wave, Square pulse Etc .

A signal which cannot be described by the Explicit mathematical representation, a table of data (or) a well-defined rule is called random/ Stochastic process.



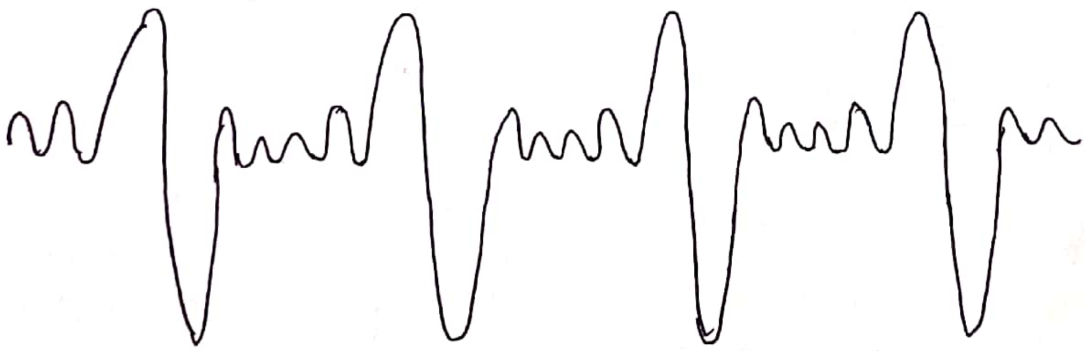
A random signal takes on one of several possible values at each time for which a signal value is defined. i.e., it is a signal about which there is uncertainty w.r.t its value at any time.

Ex: Noise generated in electronic components, Noise generated in the amplifier of a radio receiver, transmission channels, cabels etc.

Note :-

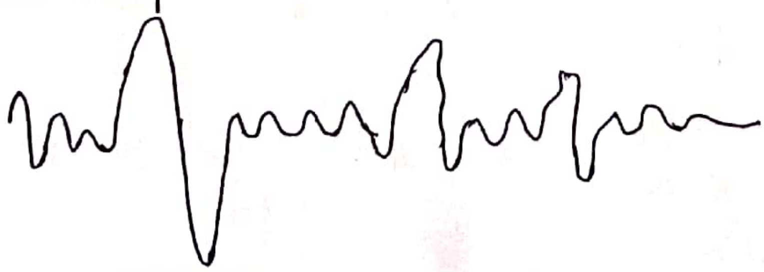
1) Signal where amplitude can be predicted before its actual occurance is said to be deterministic signal.

Ex:- ECG from good heart.



2) Signal where amplitude cannot be predicted before its actual occurance is said to be random signal.

Ex:- Failure ECG



5) Energy and power signals :-

In electrical system, a signal may be in the form of voltage (or) current. Consider, a voltage $V(t)$ exists across a resistor resulting in a current $i(t)$.

Then, the instantaneous power $p(t)$ is given by,

$$p(t) = \frac{V^2(t)}{R} = R.i^2(t)$$

The total energy expended over the time interval $t_1 \le t \le t_2$ is given by,

$$\int_{t_1}^{t_2} p(t)\,dt = \int_{t_1}^{t_2} \frac{V^2(t)}{R}\,dt = \int_{t_1}^{t_2} R.i^2(t)\,dt$$

and the average power over the time interval is,

$$\frac{1}{t_2-t_1}\int_{t_1}^{t_2} p(t)\,dt = \frac{1}{t_2-t_1}\int_{t_1}^{t_2} \frac{V^2(t)}{R}\,dt$$

The total energy of a continuous-time signal $x(t)$ is,

$$E = \underset{T\to\infty}{Lt}\int_{-T/2}^{T/2} x^2(t)\,dt = \int_{-\infty}^{\infty} x^2(t)\,dt$$

If $x(t)$ is complex then,

$$\left\langle E = \underset{T\to\infty}{Lt}\int_{-T/2}^{T/2} |x(t)|^2\,dt = \int_{-\infty}^{\infty} |x(t)|^2\,dt \right\rangle$$

The average power of a continuous-time signal $x(t)$ is given by,

$$P = \underset{T\to\infty}{Lt}\ \frac{1}{T}\int_{-T/2}^{T/2} |x(t)|^2\,dt$$

The average power of a periodic continuous-time signal $x(t)$ of fundamental period 'T' is given by,

$$\left\langle P = \frac{1}{T}\int_{-T/2}^{T/2} x^2(t)\,dt \right\rangle$$

Similarly, the total energy of a discrete-time signal $x(n)$ is given by,

$$E = \underset{N\to\infty}{Lt} \sum_{n=-N}^{N} |x(n)|^2 = \sum_{n=-\infty}^{\infty} |x(n)|^2$$

The average power of a discrete-time sequence $x(n)$ is given by,

$$P = \underset{N\to\infty}{Lt} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2$$

The average power of a periodic discrete-time signal $x(n)$ of fundamental period N is given by,

$$P = \frac{1}{N} \sum_{n=0}^{N-1} x^2(n)$$

The signal $x(t)$ (or) $x(n)$ is referred as energy signal if the total energy 'E' of the signal satisfies the condition,

$0 < E < \infty$ [i.e., E must be finite].

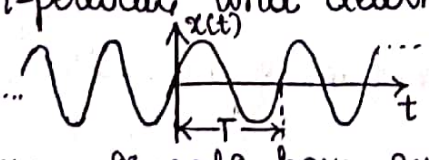
The signal $x(t)$ (or) $x(n)$ is referred as power signal if the average power 'p' of the signal satisfies the condition,

$0 < p < \infty$ [i.e., p must be finite].

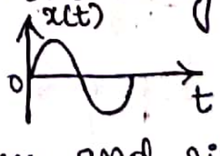
Ex:- All periodic and random signals are power signals.

Non-periodic and deterministic signals are energy signals.

Note:- CTS



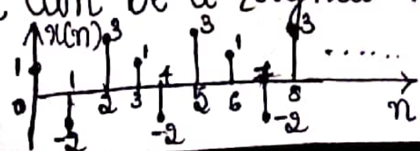
power signal (periodic)



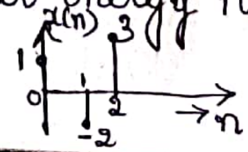
Energy signal (Aperiodic)

1. All energy signals have zero average power and finite.
2. power signal and energy signal are mutually exclusive.
3. A power signal has infinite energy and finite power.
4. There can be a signal which is neither energy nor power signal.

DTS



power signal (periodic)



Problems on Even and odd Signals :-

1. For the following Signal, check whether the Signal is odd (or) Even Signal.

a) $x(t) = \cos(w_0 t) \longrightarrow (1)$

Sol^n: put $t=-t$ in Eq(1),

$$x(-t) = \cos(w_0(-t))$$
$$x(-t) = \cos(w_0 t)$$
$$x(-t) = x(t)$$

∴ The given signal is Even Signal.

b) $x(t) = \sin(w_0 t) \longrightarrow (1)$

Sol^n: put $t=-t$ in Eq(1),

$$x(-t) = \sin(w_0(-t))$$
$$x(-t) = -\sin(w_0 t)$$
$$x(-t) = -x(t)$$

∴ The given signal is odd signal.

c) $x(n) = 2+n^2+2n^4 \longrightarrow (1)$

Sol^n: put $n=-n$ in Eq(1),

$$x(-n) = 2+(-n)^2+2(-n)^4$$
$$x(-n) = 2+n^2+2n^4$$
$$x(-n) = x(n)$$

∴ The given signal is Even Signal.

Problems on periodic and Aperiodic Signal :-

1. Check whether the following continuous signals are periodic/Aperiodic. If periodic, find the fundamental time period.

a) $x(t) = \sin\left(\frac{2\pi}{3}t\right)$

Solⁿ: Comparing with $x(t) = \sin(wt)$,

fundamental angular frequency,

$$W = \frac{2\pi}{3}$$

Fundamental period,

$$T = \frac{2\pi}{W} = \frac{2\pi}{\frac{2\pi}{3}} = \underline{\underline{3 \text{ Sec}}}$$ (rational)

$x(t)$ is a periodic signal with fundamental time period $T = 3$ Sec.

b) $x(t) = \cancel{2}\cdot\sin\left(t + \frac{\pi}{4}\right)$

Solⁿ: Comparing with $x(t) = 2\sin(Wt + \phi)$

Fundamental angular frequency, $W = 1$

Fundamental period, $T = \frac{2\pi}{W} = \underline{\underline{2\pi \text{ Sec}}}$ (rational)

∴ $x(t)$ is a periodic signal with fundamental period $T = 2\pi$ Sec.

c) $x(t) = \cos\left(t + \frac{\pi}{4}\right)$

Solⁿ:- Comparing with $x(t) = \cos\left(wt + \frac{\pi}{4}\right)$

$$\Rightarrow W = 1$$

$$T = \frac{2\pi}{W} = \frac{2\pi}{1} = \underline{\underline{2\pi \text{ Sec}}} \text{ (rational)}$$

∴ $x(t)$ is a periodic signal with $T = \underline{\underline{2\pi \text{ Sec}}}$.

d) $x(t) = \cos^2(t)$

Solⁿ:- $x(t) = \frac{1 + \cos(2t)}{2}$

$$\cos^2\theta = \frac{1 + \cos 2\theta}{2}$$

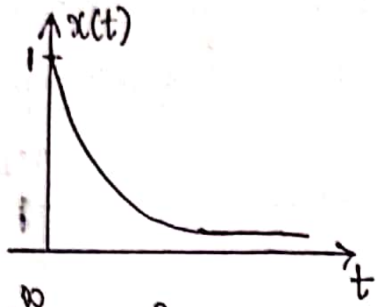
$$\sin^2\theta = \frac{1 - \cos 2\theta}{2}$$

Problems on Energy and power Signals:-

I. Check whether the following signals are Energy (or) power signals. Find the Corresponding energy (or) power associated with the Signal.

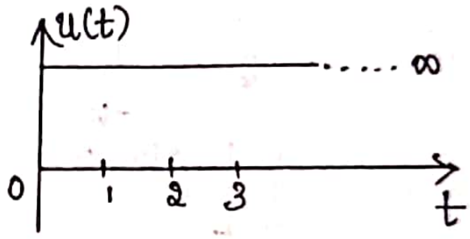
1) $x(t) = e^{-at}u(t)$ (or) $x(t) = e^{-at}$; $0 \le t \le \infty$; $a > 0$

Sol^n:



Given signal is non-periodic signal. Hence, it can be Energy signal.

$u(t) = 1$ for $t \ge 0$ (or) $0 \le t \le \infty$
0 ; otherwise



$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

$$= \int_{0}^{\infty} (e^{-at})^2 dt$$

$$= \int_{0}^{\infty} e^{-2at} . dt$$

$$= \left. \frac{e^{-2at}}{-2a} \right|_0^{\infty}$$

$$= \frac{-1}{2a} \left[e^{-\infty} - e^{0} \right]$$

$$= \frac{-1}{2a} (0 - 1)$$

$$\boxed{E = \frac{1}{2a} \text{ joules}}$$

Since 'E' is finite. $\therefore$, It is Energy signal.

2) $x(t) = e^{2t} u(-t)$

Sol^n: $E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{0} (e^{2t} . 1)^2 dt$

$u(-t) = 1$ for $-\infty \le t \le 0$
0 ; otherwise

Given signal is Non-periodic signal. $\therefore$ it can be Energy signal.

Elementary Signals :-

Elementary signals are the basic building blocks for constructing more complex signals. These signals are used to test the system.

Some of the elementary signals are :

a) Exponential Signals
b) Sinusoidal Signals
c) Exponentially damped Sinusoidal Signals
d) unit Step function
e) unit impulse function
f) unit ramp function.
g) Rectangular function.
h) Triangular function
i) Signum function
j) Sinc function.

a) Exponential Signals :-

Continuous-time Signal :-

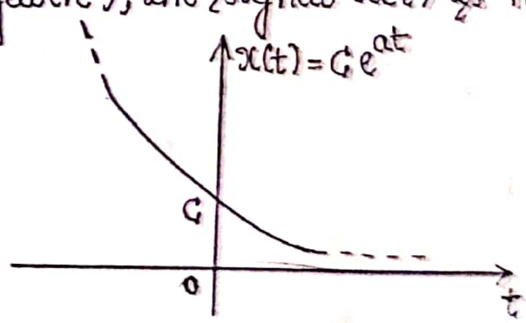
A real Exponential Continuous-time signal is given by,

$$x(t) = C\, e^{at}$$

Where both 'C' and 'a' real constant.

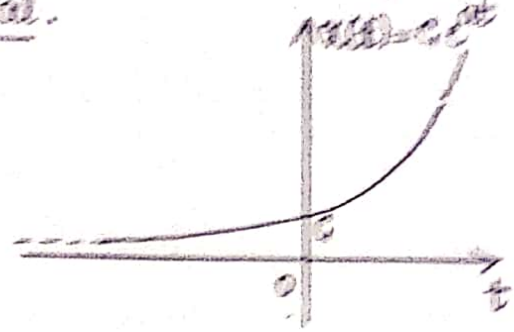
'C' is known as the amplitude of the exponential signal at $t=0$.

If $a<0$ (i.e., 'a' is negative), the signal $x(t)$ is known as decaying exponential signal.



Real decaying exponential signal (i.e., $a<0$).

If $a>0$ (i.e., 'a' is positive), the signal $x(t)$ is called growing exponential signal.



Real growing exponential signal (i.e., $a>0$)

Note :- If "C" (or) "a" (or) both are complex numbers, then $x(t)$ is known as Continuous-time Complex Exponential Signal.

Consider $C=1$ and 'a' is imaginary i.e., $x(t)=e^{j\omega_0 t}$.

Discrete-time Signal :-

A real exponential discrete-time signal (or) sequence is given by,

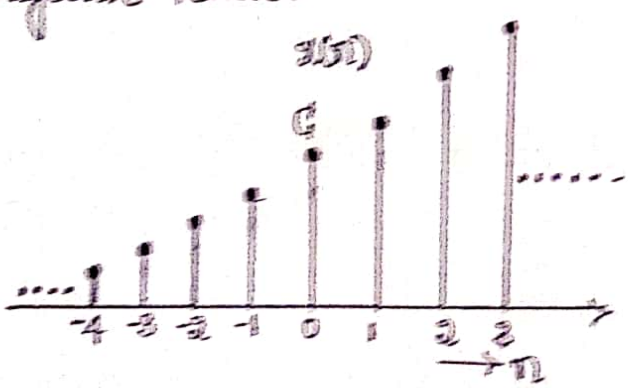
$$x(n) = C\alpha^n$$

where $\alpha = e^{\beta}$

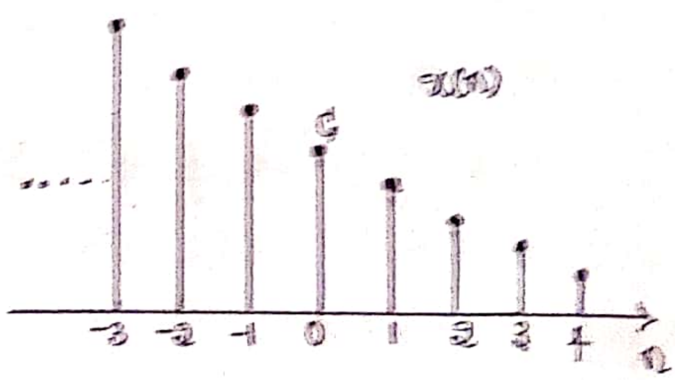
and C, α and β are real constants.

'C' is known as the amplitude of the sequence at $n=0$.

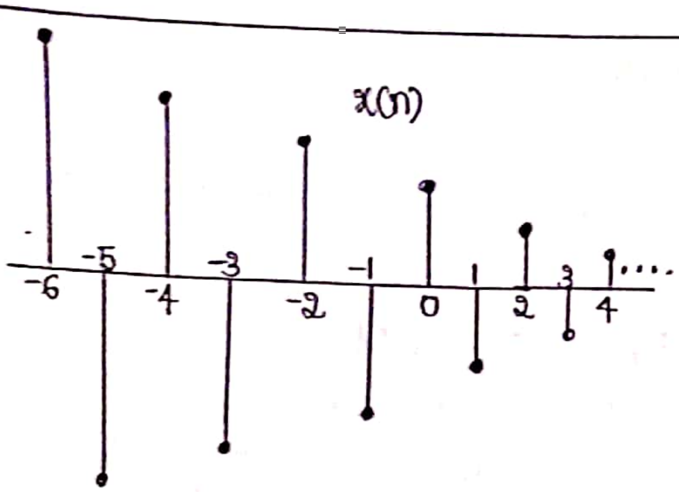
If $|\alpha|<1$, the signal decays exponentially. If $\alpha<0$ (i.e., α is negative), then the sign of $x(n)$ alternates i.e., when 'n' is positive, $x(n)$ has positive value and when 'n' is negative, $x(n)$ has negative value.



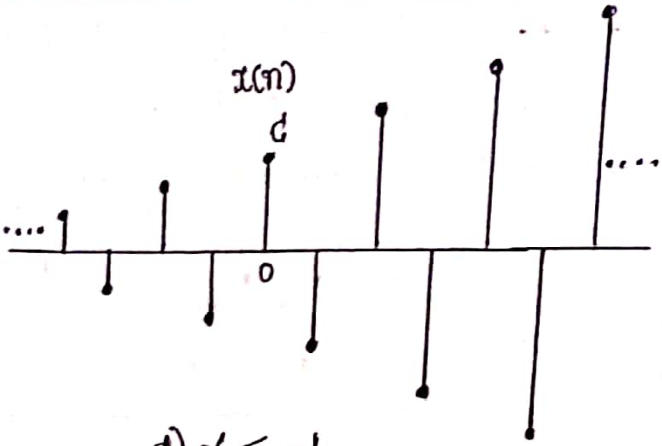
a) $\alpha>1$



b) $0<\alpha<1$



c) $-1 < \alpha < 0$



d) $\alpha < -1$

Note:-

If 'C' (or) 'α' (or) both are complex numbers, then $x(n)$ is known as discrete-time complex exponential sequence.

Consider $C = 1$ and β is purely imaginary. i.e., $x(n) = e^{j\Omega_0 n}$.

b) Sinusoidal Signals :-

Continuous-time Signal :-

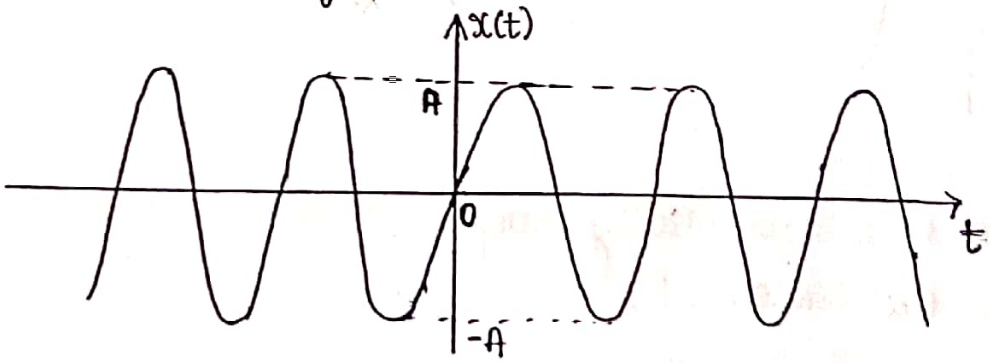
A sinusoidal signal is given by,

$$x(t) = A \sin(\omega_0 t + \phi)$$

where $W_0 = 2\pi f_0$ = angular frequency (rad/sec)

$f_0 \rightarrow$ linearly frequency (Hz)

$\phi \rightarrow$ phase shift (radians).



Discrete-time Signal :-

A discrete-time sinusoidal signal is given by,

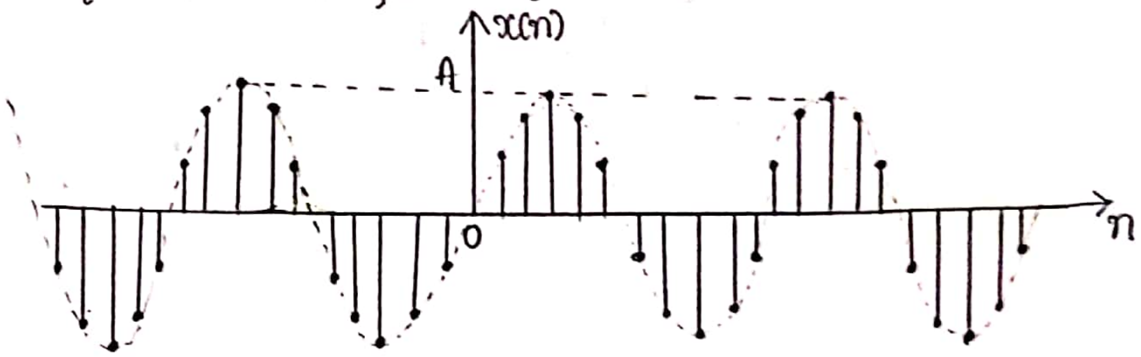
$$x(n) = A \sin(\Omega_0 n + \phi)$$

where 'A' is maximum value of $x(n)$.

$\Omega_0 \rightarrow$ angular frequency in rad.

$\phi \rightarrow$ phase angle.

If 'n' is dimensionless, both Ω_0 and ϕ are measured in radians.



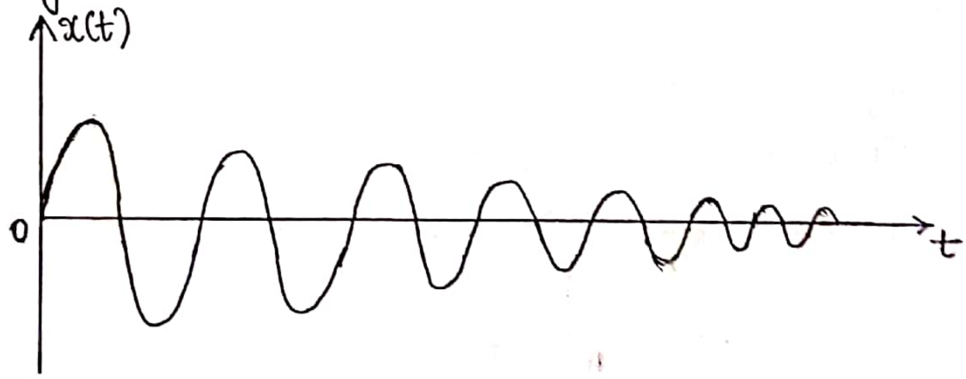
c) Exponential damped sinusoidal signals :—

Continuous-time signal :—

An exponentially damped sinusoidal signal is given by,

$x(t) = e^{-at} \sin wt$; where $a > 0$.

As 't' increases, the amplitude of sinusoidal oscillation decreases exponentially and approaches zero at $t \rightarrow \infty$.



Discrete-time signal :—

A discrete-time exponentially damped sinusoidal sequence is given by,

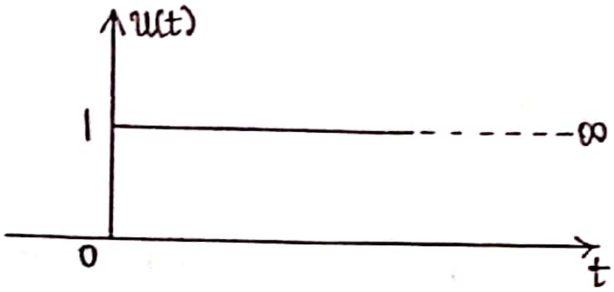
$x(n) = C\alpha^n \sin(\Omega n + \phi)$; $0 < |\alpha| < 1$

The value of x(n) decreases as 'n' increases.

d) Unit Step function : Continuous-time Signal ($u(t)$) :-

The Continuous-time unit step function is defined as,

$$u(t) = \begin{cases} 1 \; ; \; t \geq 0 \text{ (or) } 0 \leq t \leq \infty \\ 0 \; ; \text{ otherwise (or) } t < 0 . \end{cases}$$



properties:

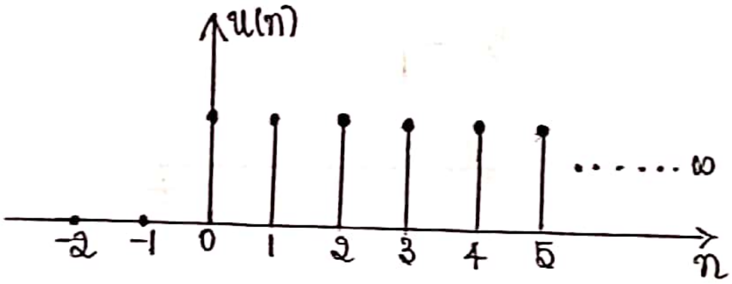
1) $u(t-t_0) = [u(t-t_0)]^2 = [u(t-t_0)]^K$ where K is any $+^{ve}$ integer. This property is based on the relations $(0)^K = 0$ and $(1)^K = 1$, $K = 1, 2, 3, \ldots$

2) $u(at - t_0) = u\left(t - \frac{t_0}{a}\right)$; $a \neq 0$ (Time-Scaling property).

Discrete-time Signal ($u(n)$) :-

A discrete-time unit step sequence is defined as,

$$u(n) = \begin{cases} 1 \; ; \; n \geq 0 \text{ (or) } 0 \leq n \leq \infty \\ 0 \; ; \text{ otherwise (or) } n < 0 \end{cases}$$



Note :- It is used as best test signal.

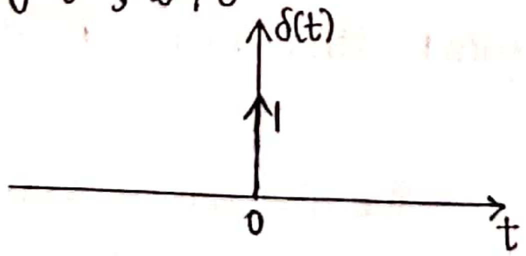
Area under unit step-function is unity.

e) Unit Impulse function :

Continuous-time Signal :- $\delta(t)$

The Continuous-time unit impulse function $\delta(t)$ is defined as,

$$\delta(t) = \begin{cases} 1 \; ; \; t = 0 \\ 0 \; ; \; t \neq 0 \end{cases}$$



Discrete-time Signal ($u(n)$) :-

A discrete-time unit impulse sequence is defined as,

$$\delta(n) = \begin{cases} 1 \; ; \; n = 0 \\ 0 \; ; \; n \neq 0 \end{cases}$$

f) unit ramp function :-

Continuous-time signal ($x(t)$):-

A ramp function is defined as,

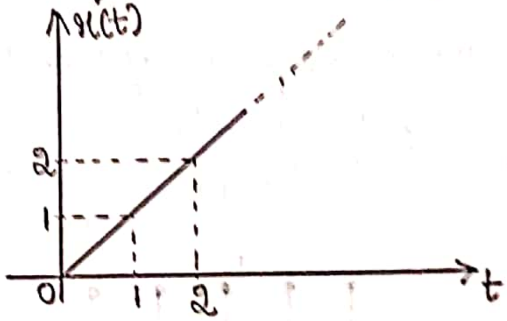
$$x(t) = \begin{cases} t \;;\; t \geq 0 \\ 0 \;;\; \text{otherwise} \end{cases} \quad \text{(or)} \quad x(t) = \begin{cases} t\,u(t) \text{ for } t \geq 0 \\ 0 \;;\; \text{otherwise} \end{cases}$$

Ramp function is the integral of the unit-step function $u(t)$.

i.e.,

$$x(t) = \int u(t)\,dt = \int 1\,dt = t$$

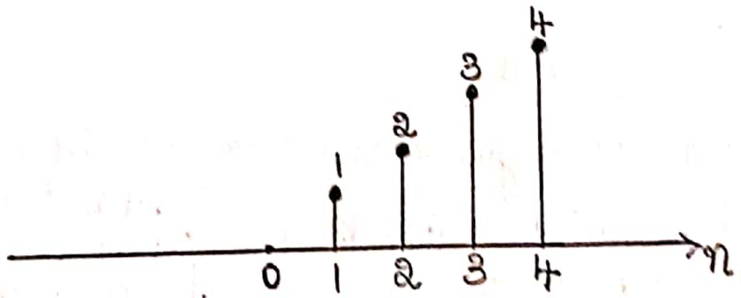
$$u(t) = \frac{d}{dt}\,x(t).$$



Discrete-time signal ($x(n)$):-

A discrete-time ramp sequence is defined as,

$$x(n) = \begin{cases} n \;;\; n \geq 0 \\ 0 \;;\; n<0 \end{cases} \quad \text{(or)} \quad x(n) = \begin{cases} n\,u(n) \;;\; n \geq 0 \\ 0 \;;\; n<0. \end{cases}$$



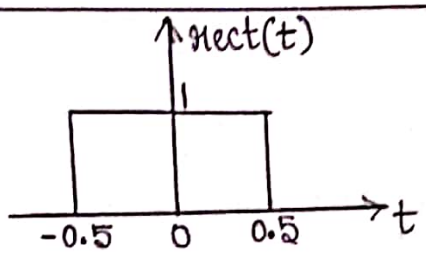
g) pulse signal :-

i) Rectangular function :-

Continuous-time signal ($rect(t)$):-

A continuous-time rectangular function is defined as,

$$rect(t) = \begin{cases} 1 \;;\; |t| \leq 0.5 \\ 0 \;;\; \text{otherwise} \end{cases} \quad \text{(or)} \quad rect(t) = \begin{cases} 1 \;;\; -0.5 \leq t \leq 0.5 \\ 0 \;;\; \text{otherwise}. \end{cases}$$



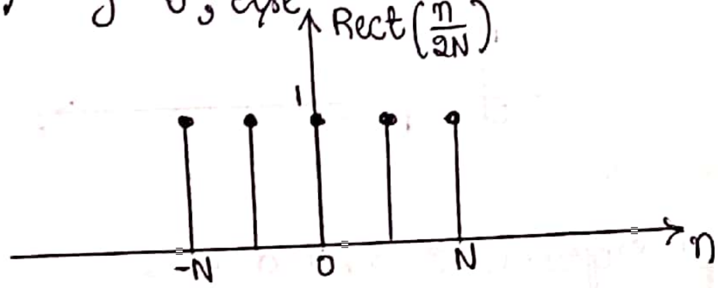
Height = 1
Width = 1
Area = 1

The Signal $x(t) = \text{rect}\left(\frac{t-b}{a}\right)$ describes a rectangular pulse of width 'a' centred at $t = b$.

Discrete-time Signal ($\text{rect}\left(\frac{n}{2N}\right)$):—

A discrete-time rectangular function is defined as,

$$\text{Rect}\left(\frac{n}{2N}\right) = \begin{cases} 1 ; & |n| \leq N \\ 0 ; & \text{else} \end{cases}$$



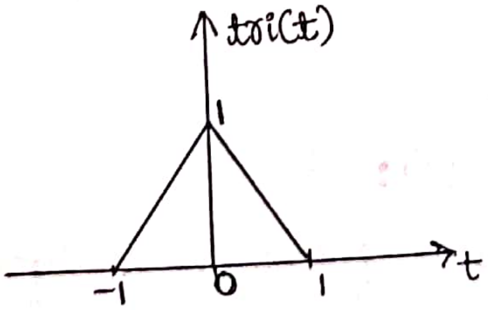
The Signal $\text{rect}\left(\frac{n}{2N}\right)$ has $2N+1$ unit samples over $-N \leq n \leq N$.

ii) Triangular function:—

Continuous-time Signal ($\text{tri}(t)$):—

A Continuous-time triangular function is defined as,

$$\text{tri}(t) = \begin{cases} 1 - |t| ; & |t| \leq 1 \\ 0 ; & \text{otherwise} \end{cases} \quad \text{(or)} \quad \text{tri}(t) = \begin{cases} 1 + t ; & -1 \leq t \leq 0 \\ 1 - t ; & 0 \leq t \leq 1 \\ 0 ; & \text{otherwise} \end{cases}$$



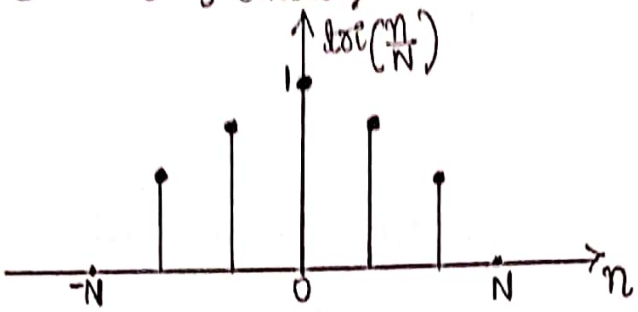
Height = 1
Area = 1
Width = 2

The Signal $y(t) = \text{tri}\left(\frac{t-b}{a}\right)$ describes a triangular pulse of width '2a' centered at $t = b$.

Discrete-time signal ($tri(\frac{n}{N})$) :-

A discrete-time triangular function is defined as,

$$tri\left(\frac{n}{N}\right) = \begin{cases} 1-\frac{|n|}{N} & ; |n| \le N \\ 0 & ; \text{otherwise} \end{cases}$$



The signal $x(n) = tri\left(\frac{n}{N}\right)$ has $(2N+1)$ samples over $-N \le n \le N$, with $x(N)$ and $x(-N)$ samples being zero.

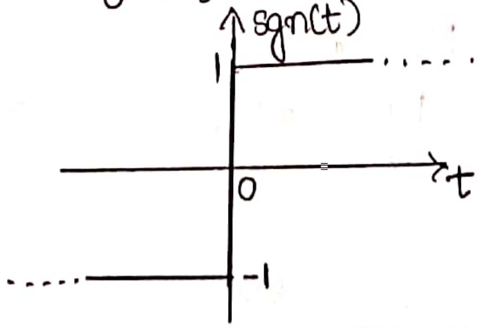
Note :- Rectangular and triangular pulses satisfies the symmetry property.

h) Signum function :-

Continuous-time signal ($sgn(t)$) :-

A continuous-time signum function is defined as,

$$sgn(t) = \begin{cases} 1 & ; t > 0 \\ 0 & ; t = 0 \\ -1 & ; t < 0 \end{cases}$$



Discrete-time signum function ($sgn(n)$) :-

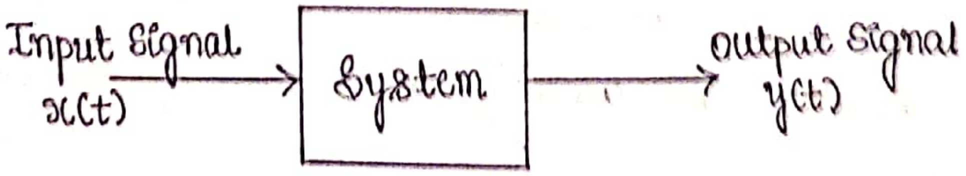
A discrete-time signum function is defined as,

$$sgn(n) = \begin{cases} 1 & ; n > 0 \\ 0 & ; n = 0 \\ -1 & ; n < 0 \end{cases}$$

Operations on the Signals

Basic operations on signals :—

Mathematical operations performed on signals by systems can be classified as operations on dependent variables and independent variables.



Dependent variable corresponds to the amplitude (or) value of the signal but the independent variable is time 't' (or) 'n' for continuous-time and discrete-time signal respectively.

a) Operations performed on the dependent variables :—

i) Amplitude Scaling :—

Let $x(t)$ be a continuous-time signal. Then the signal $y(t)$ is amplitude scaling of $x(t)$ is defined as,

$$\boxed{y(t) = c\, x(t)}$$

Where 'c' is scaling factor.

The signal $y(t)$ is obtained by multipling the value (amplitude) of $x(t)$ by scalar 'c' at all 't'.

Similarly, let $x(n)$ be a discrete-time signal. Then, the signal $y(n)$ is amplitude scaling of $x(n)$ is defined as,

$$\boxed{y(n) = c\, x(n)}$$

where 'c' is a scaling factor.

The signal $y(n)$ is obtained by multiplying the value of $x(n)$ by scalar 'c' at all 'n'. Ex: Amplifier, Attenuator.

If resistor (C) is amplitude scaling when $x(t)$ is a current, then $y(t)$ is the output voltage.

ii) Addition :-

Let $x_1(t)$ and $x_2(t)$ denote a pair of continuous-time signals. Then the signal, $y(t)$ is obtained by the addition of $x_1(t)$ and $x_2(t)$ for all 't'.

$$\therefore \boxed{y(t) = x_1(t) + x_2(t)}.$$

Similarly, let $x_1(n)$ and $x_2(n)$ are discrete-time signals. Then, the signal, $y(n)$ is obtained by adding the values of $x_1(n)$ and $x_2(n)$ for all 'n'.

$$\therefore \boxed{y(n) = x_1(n) + x_2(n)}$$

Ex:- An audio mixture i.e., which combines music and voice signals.

iii) Multiplication :-

Let $x_1(t)$ and $x_2(t)$ denote a pair of continuous-time signals. Then, the signal $y(t)$ is obtained by multiplication of $x_1(t)$ and $x_2(t)$ for all 't'.

$$\therefore \boxed{y(t) = x_1(t) \cdot x_2(t)}$$

Similarly, let $x_1(n)$ and $x_2(n)$ are discrete-time signals. Then, the signal $y(n)$ is obtained by taking the product of $x_1(n)$ and $x_2(n)$ for all 'n'.

$$\therefore \boxed{y(n) = x_1(n) \cdot x_2(n)}$$

Ex:- An AM radio signal, in which it consists of audio signal and sinusoidal signal.

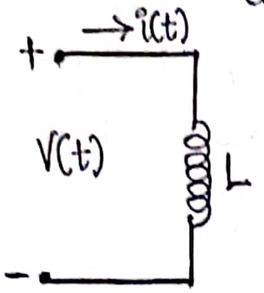
iv) Differentiation :-

Let $x(t)$ be a continuous-time signal. Then the differentiation of $x(t)$ with respect to time 't' is defined as,

$$\boxed{y(t) = \frac{d}{dt} x(t)}$$

An inductor performs differentiation i.e.,

$$V(t) = L \frac{d}{dt} i(t)$$



$i(t) \rightarrow$ Current through 'L'

$L \rightarrow$ Inductor

$V(t) \rightarrow$ developed Voltage.

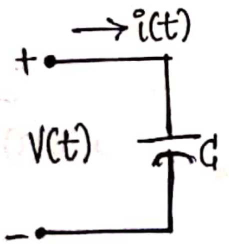
v) Integration :-

Let $x(t)$ be a continuous-time signal. Then, the integration of $x(t)$ with respect to time 't' is defined as,

$$y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$$

A capacitor performs integration.

$$V(t) = \frac{1}{C} \int_{-\infty}^{t} i(\tau)\, d\tau$$



$i(t) \rightarrow$ Current through C

$C \rightarrow$ Capacitor

$V(t) \rightarrow$ Developed Voltage.

b) Operations performed on the independent Variables :-

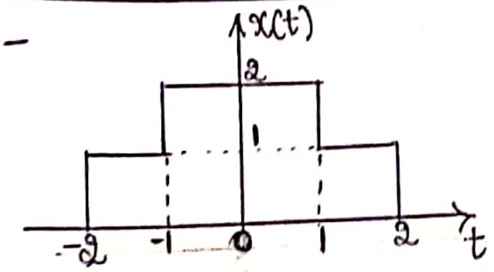
i) Time Scaling :-

Let $x(t)$ be a continuous-time signal. The signal $y(t)$ obtained by scaling the independent variable 't' by a factor 'a' is given by,

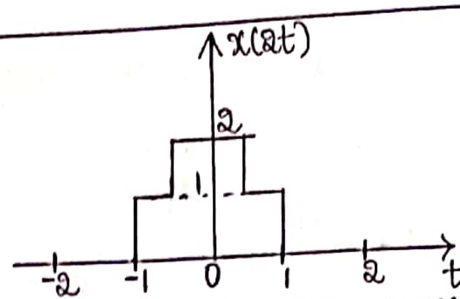
$$\boxed{y(t) = x(at)}$$

If $a > 1$, the signal $y(t)$ is a compressed version of $x(t)$ and if $0 < a < 1$, the signal $y(t)$ is an expanded version of $x(t)$.

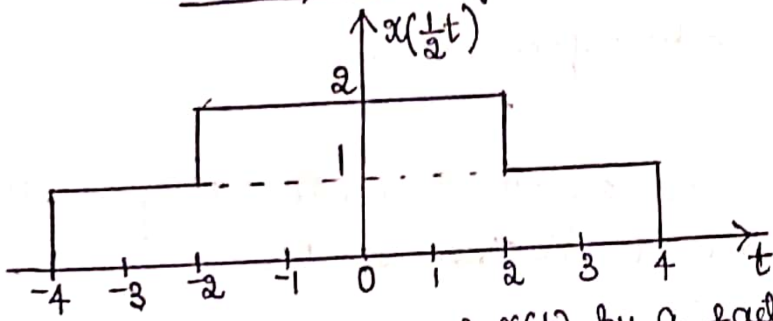
Ex :-



Continuous-time signal $x(t)$



Compressed version of $x(t)$ by a factor 2



Expanded version of $x(t)$ by a factor 2

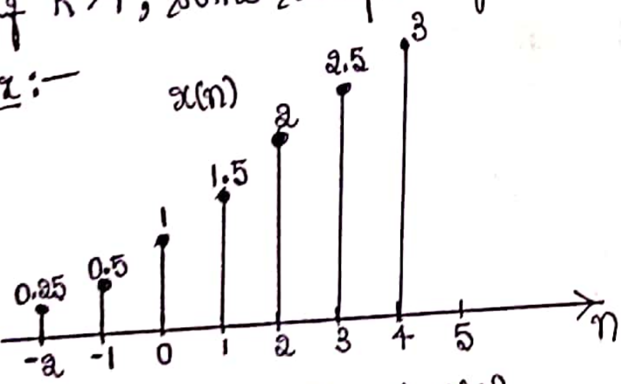
Similarly, for discrete-time sequence,

$$y(n) = x(Kn) \quad ; K > 0$$

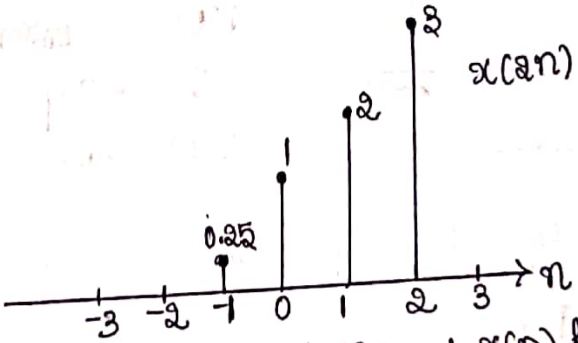
where 'K' is an integer.

If $K > 1$, some samples of $x(n)$ would be lost.

Ex :-



Discrete-time signal $x(n)$



Compressed version of $x(n)$ by a factor 2

(ii) Time-shifting :-

Let $x(n)$ be a continuous-time signal. Then the signal, $y(t)$ is time-shifted version of $x(t)$ is defined as,

$$\boxed{y(t) = x(t - t_0)}$$

where 't_0' is the time shift.

put $t=0$ in Eq(1),

$$y(0) = x(-b) \longrightarrow (2)$$

put $t = b/a$ in Eq(1),

$$y(b/a) = x(0) \longrightarrow (3)$$

Once we obtain $y(t)$ from $x(t)$ by performing time shifting and time scaling operation, it must satisfy Eq(2) and Eq(3).

This is possible only if the time shifting operation is performed first on $x(t)$, which yields an intermediate signal $v(t)$ is given by,

$$v(t) = x(t-b)$$

Next, the time scaling operation is performed on $v(t)$ to obtain $y(t)$.

i.e., $y(t) = v(at) = x(at-b)$.

Note :-

precedence : Time shifting
Time scaling
Time folding (Reflection).

Systems :-

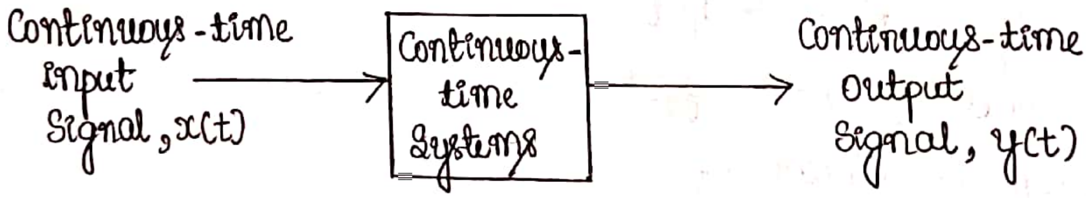
A System is a set of Elements (or) functional blocks that are connected together and produce an output in response to an input signal (or) System is an entity that manipulates one (or) more signal to accomplish a function, thereby yielding new signals.
Ex: Computer.

Classification of Systems :-

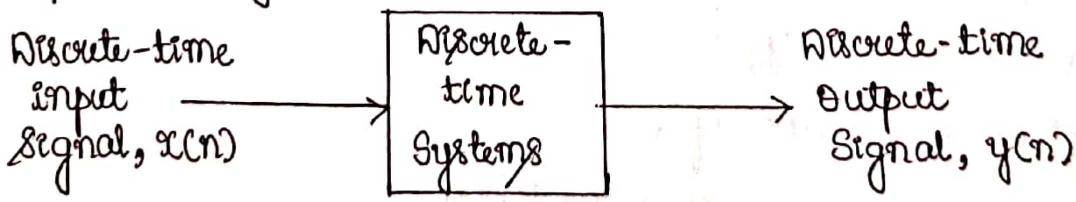
There are two types of systems :

i) Continuous-time Systems
ii) Discrete-time systems.

Continuous-time (CT) systems handle continuous-time signals.
Ex: Analog filters, amplifiers, attenuators, analog transmitters and receivers etc.



Discrete-time (DT) systems handle discrete-time signals.
Ex: Computers, printers, microprocessors, memories, Shift registers etc.
They operate only on discrete-time signals.



Continuous as well as discrete time systems can be further classified based on their properties. These properties are as follows :

i) Dynamicity property : Static and dynamic systems.
ii) Shift invariance : Time invariant and time-variant systems.
iii) Linearity property : Linear and non-linear systems.

iv) Causality property : Causal and non-causal systems.

v) Stability property : Stable and unstable systems.

vi) Invertibility property : Invertible and non-invertible systems.

Properties of Systems :

i) Linearity :-

A System is said to be linear if it satisfies the principle of superposition. i.e., if an input consists of the weighted sum of several signals, then the output is the weighted sum of the responses of the system to each of those signals.

Let the input $x_1(t)$ applied to a continuous-time system results in output $y_1(t)$ and another input $x_2(t)$ results in output $y_2(t)$. Then, if the system gives output $y_1(t)+y_2(t)$ for the input $x_1(t)+x_2(t)$, the system is said to be linear.

Alternatively,

$$\text{if } x_1(t) \longrightarrow y_1(t)$$
$$\text{and } x_2(t) \longrightarrow y_2(t)$$

Then, the system is linear if,

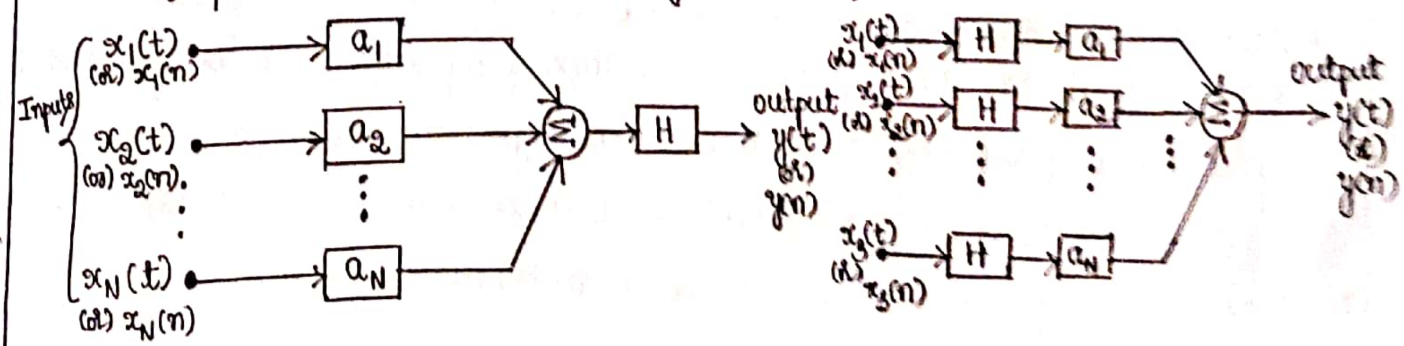
$$a\,x_1(t) + b\,x_2(t) \longrightarrow a\,y_1(t) + b\,y_2(t)$$

Similarly, Consider a discrete-time system with

$$x_1(n) \longrightarrow y_1(n)$$
$$\text{and } x_2(n) \longrightarrow y_2(n)$$

then the system is linear if,

$$a\,x_1(n) + b\,x_2(n) \longrightarrow a\,y_1(n) + b\,y_2(n).$$



A System is said to be non-linear if it does not satisfies the principle of superposition.

Note:-

i) $y(t) = H\left\{\sum_{i=1}^{N} a_i x_i(t)\right\} = \sum_{i=1}^{N} a_i H\{x_i(t)\}$.

ii) Time-Invariance :-

A time-invariant system is one for which a time shift of the input signal causes a corresponding time shift in the output signal. The shift may be advance (or) delay.

Specifically, suppose that a continuous-time system gives output $y(t)$ for an input $x(t)$, then the system is said to be time-invariant if the input $x(t-t_0)$ gives output $y(t-t_0)$.

i.e., If $x(t) \longrightarrow y(t)$

then the system is time-invariant if,

$$x(t-t_0) \longrightarrow y(t-t_0)$$

Similarly, a discrete-time system with,

$$x(n) \longrightarrow y(n)$$

is said to be time-invariant if,

$$x(n-n_0) \longrightarrow y(n-n_0).$$

A System is said to be time-variant if the input $x(t-t_0)$ does not produce output $y(t-t_0)$.

i.e., $x(t-t_0) \neq y(t-t_0)$.

iii) Memory :-

A system is said to be memory (dynamic), if its output signal $y(t)/y(n)$ depends on past (or) future value of the input signal $x(t)/x(n)$.

A system is said to be memoryless (static), if its output signal $y(t)/y(n)$ depends only on the input signal $x(t)/x(n)$ at the same value of 't'/'n'.

iv) Causality :-

A System is said to be causal if the present value of the output y(t)/y(n) depends only on the past and/or present value of the input x(t)/x(n).

A System is said to be non-causal if the present value of the output y(t)/y(n) depends on future values of the input signal, x(t)/x(n).

Ex: $y(n) = \frac{1}{3}[x(n) + x(n-1) + x(n-2)]$ ⟶ causal System

$y(n) = \frac{1}{3}[x(n+1) + x(n) + x(n-1)]$ ⟶ Non-causal System.

V) Stability :-

A System is said to be bounded input-bounded output (BIBO) stable if and only if every bounded (finite value) input results in a bounded (finite value) output. The output of such a system does not diverge if the input does not diverge.

The Condition for BIBO is,

$$|y(t)| \leq M_y < \infty \text{ ; for all 't'.}$$

whenever, the input signal x(t) satisfy the condition

$$|x(t)| \leq M_x < \infty \text{ ; for all 't'}$$

Both M_x and M_y represent some finite positive numbers.

Similarly, a discrete-time System is stable if the output y(n) satisfies the Condition,

$$|y(n)| \leq M_x < \infty \text{ ; for all 'n'}$$

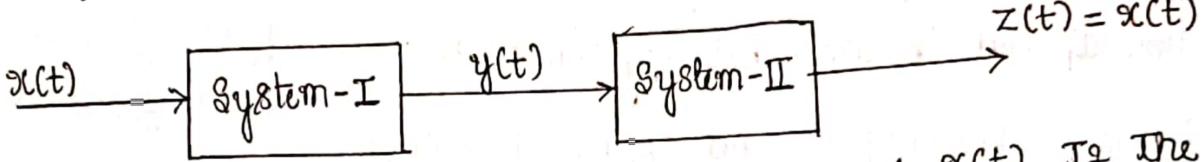
whenever, the input signal x(n) satisfy the condition

$$|x(n)| \leq M_y < \infty \text{ ; for all 'n'.}$$

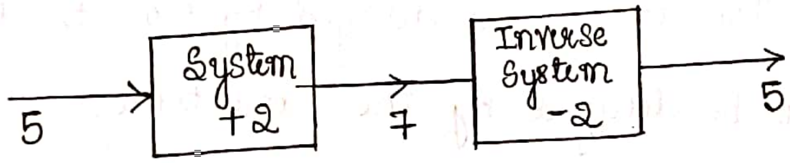
vi) Invertibility :-

A System is said to be invertible if the input of the system can be recovered from the system output i.e., if inverse system exists it is invertible.

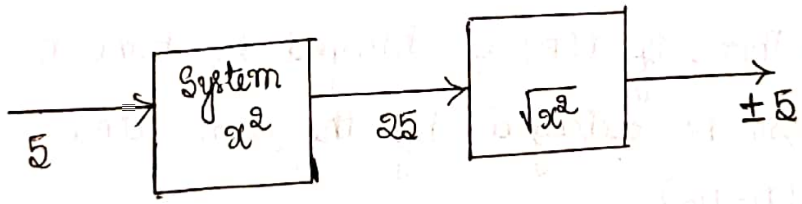
Consider, a cascade connection of continuous-time system.



The System-I gives the output y(t) for an input x(t). If the input x(t) to the System-I can be recovered from y(t) by connecting System-II in cascade with System-I, then System-I is said to be invertible and the System-II is called inverse System.



Invertible



Non-Invertible

3) Memory :-

If output of the continuous-time system (or) discrete-time system depends upon the present input only, then it is called static (or) memoryless (or) instantaneous system.

Ex: $y(n) = 10x(n)$, $y(n) = 15.x^2(n) + 10x(n)$.

A system is said to be dynamic if the output depends upon the past values of input also.

Ex: $y(n) = x(n) + x(n-1)$

$$y(n) = \sum_{K=0}^{4} x(n-K) = x(n) + x(n-1) + x(n-2) + x(n-3) + x(n-4).$$

4) Causal :

The system is said to be casual if its output at any time depends upon present and past inputs only.

i.e., $y(t_0) = f[x(t) ; t \leq t_0]$. for CTS.

$$y(n) = f[x(K) ; K \leq n]$$

Thus, $y(n)$ is a function of $x(n), x(n-1), x(n-2), x(n-3), \ldots$ etc for causal system and $y(n)$ is a function of $x(n+1), x(n+2), x(n+3) \ldots$ etc for non-causal system (its output depends upon future inputs also).

5) Stability :-

When every bounded input produces bounded output, then the system is called bounded input bounded output (BIBO) stable.

CT input : $|x(t)| \leq M_x < \infty$ CT input : $|y(t)| \leq M_y < \infty$

DT input : $|x(n)| \leq M_x < \infty$ DT input : $|y(n)| \leq M_y < \infty$

If the system produces unbounded output for bounded input, then it is unstable.

6) Invertability :

A system is said to be invertible if there is unique output for every unique input. S/m is H, then its inverse S/m is H^{-1}. Then $HH^{-1} = 1$.

Problems on Systems

I Determine whether the systems are i) Linear ii) Time-invariant iii) Memory iv) Causal v) Stable.

1) $y(t) = x(t/2)$

Sol^n:

i) Linearity :-

$y_1(t) = f[x_1(t)] = x_1(t/2)$

$y_2(t) = f[x_2(t)] = x_2(t/2)$

$y_3(t) = a\, y_1(t) + b\, y_2(t)$

$y_3(t) = a\, x_1(t/2) + b\, x_2(t/2) \longrightarrow (1)$

$y_3'(t) = f[a.x_1(t) + b\, x_2(t)]$

$y_3'(t) = a\, x_1(\frac{t}{2}) + b\, x_2(t/2) \longrightarrow (2)$

$\therefore y_3(t) = y_3'(t)$

Hence, this is linear system.

ii) Time-invariant :-

$y(t, t_0) = f[x(t - t_0)]$

$y(t, t_0) = x\left(\frac{t}{2} - t_0\right) \longrightarrow (1)$

$y(t - t_0) = x\left(\frac{t - t_0}{2}\right) \longrightarrow (2)$

$y(t, t_0) \neq y(t - t_0)$

Hence, this is time-variant S/m

iii) Memory :-

@$t = 0$, $y(0) = x(0)$; present

@$t = 1$, $y(1) = x(1/2)$; past

@$t = -1$, $y(-1) = x(-1/2)$; future

Hence, this is memory S/m. $\because$ it depends on past and future value of the input.

iv) Causal :-

@$t = 0$, $y(0) = x(0)$; present

@$t = 1$, $y(1) = x(1/2)$; past

@$t = -1$, $y(-1) = x(-1/2)$; future

Hence, this is a non-causal S/m. $\because$ it depends on the future value of the input.

v) Stable : Let $|x(t)| < M_x$, then $|y(t)| = |x(t/2)| < M_x$.

$|x(t)| \leq M_x < \infty$

$|y(t)| \leq M_y \leq \infty$

Since, it obeys BIBO. It is a stable S/m.

11) $y(t) = x^2(t)$

Sol^n :-

i) Linearity :-

$y_1(t) = f[x_1(t)] = x_1^2(t)$

$y_2(t) = f[x_2(t)] = x_2^2(t)$

$y_3(t) = a\,y_1(t) + b\,y_2(t)$

$y_3(t) = a\,x_1^2(t) + b\,x_2^2(t) \rightarrow (1)$

$y_3'(t) = f[a\,x_1(t) + b\,x_2(t)]$

$y_3'(t) = [a\,x_1(t) + b\,x_2(t)]^2 \rightarrow (2)$

$\therefore y_3(t) \neq y_3'(t)$

It is a linear S/m.

ii) Time-invariant :-

$y(t, t_0) = f[x(t-t_0)] = x^2(t-t_0) \rightarrow (1)$

$y(t-t_0) = x^2(t-t_0) \longrightarrow (2)$

$\therefore y(t, t_0) = y(t-t_0)$

It is a time-invariant S/m.

iii) Causal :-

@t=0, $y(0) = x^2(0)$; present

@t=1, $y(1) = x^2(1)$; present

@t=-1, $y(-1) = x^2(-1)$; present

$\therefore$ It is a causal S/m, since present o/p's depends only on present i/p's.

iv) Memory :-

@t=0, $y(0) = x^2(0)$; present

@t=1, $y(1) = x^2(1)$; present

@t=-1, $y(-1) = x^2(-1)$; present

It is a memoryless S/m.

v) Stability :-

let $|x(t)| \leq M_x < \infty$, then

$|y(t)| = |x^2(t)| \leq M_y < \infty$

It obeys BIBO. Hence, it is a stable S/m.

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Module-3

System Interconnection and System properties in terms of impulse response

Properties of impulse response ($h(n)$) :-

Based on the impulse response, LTI system can be memoryless, causality, stability, invertibility etc.

1) Memoryless :-

Consider a LTI system,

$$y(n) = x(n) * h(n)$$

Since convolution is commutative, hence

$$y(n) = h(n) * x(n)$$

$$= \sum_{K=-\infty}^{\infty} h(K)\, x(n-K)$$

$$y(n) = \ldots\ldots + h(-2)\,x(n+2) + h(-1)\,x(n+1) + h(0)\,x(n) + h(1)\,x(n-1) + h(2)\,x(n-2) + \ldots\ldots$$

A system is memoryless if its output depends only on the present values of the input otherwise memory.

Here, the present values of the input $x(n)$ associated with $h(0)$

$\therefore$ A LTI sym characterised by a impulse response $h(n)$ to be memoryless if $\ldots\ldots = h(-2) = h(-1) = 0 = h(1) = h(2) = \ldots\ldots$

i.e., $h(n) = 0$ for $n \neq 0$

$$\boxed{h(n) = C\,\delta(n)}$$

Similarly, for a continuous LTI system characterized by impulse response $h(t)$ to be memoryless if $\boxed{h(t) = C\,\delta(t)}$

2) Causal :-

Consider a LTI system,

$$y(n) = x(n) * h(n)$$

Since convolution is commutative, hence

$$y(n) = h(n) * x(n)$$

$$y(n) = \sum_{K=-\infty}^{\infty} h(K)\,x(n-K)$$

$$= \ldots\ldots + h(-2)\,x(n+2) + h(-1)\,x(n+1) + h(0)\,x(n) + h(1)\,x(n-1) + h(2)\,x(n-2) + \ldots\ldots$$

A system to be causal, if the output depends on the present and/or past values of the inputs, otherwise it is non-causal.
Here the present and past values of the input $x(n)$, $x(n-1)$, $x(n-2)$, ... are associated with $h(0)$, $h(1)$, $h(2)$, ... respectively.
Whereas, the future values of the input $x(n+1)$, $x(n+2)$, $x(n+3)$, ... are associated with $h(-1)$, $h(-2)$, $h(-3)$, ... respectively.

$\therefore$ A LTI system characterised by a impulse response, $h(n)$ to be causal if

$$h(-1) = 0 = h(-2) = h(-3) = \ldots\ldots$$

i.e., $\boxed{h(n) = 0 \text{ for } n < 0}$

Similarly, for a continuous-time LTI system to be causal if

$\boxed{h(t) = 0 \text{ for } t < 0}$

3) Stable :-

Consider a LTI system,

$$y(n) = x(n) * h(n)$$

Since convolution is commutative, hence

$$y(n) = h(n) * x(n)$$

$$= \sum_{K=-\infty}^{\infty} h(K)\,x(n-K)$$

Consider,

$$|y(n)| = \left| \sum_{K=-\infty}^{\infty} h(K)\,x(n-K) \right|$$

$$|y(n)| \le \sum_{K=-\infty}^{\infty} |h(K)|\,|x(n-K)|$$

; using triangular inequality
sum of magnitude $\ge$ magnitude of sum.

A system to be stable if for every bounded input there exist bounded output.

If $|x(n)| \le M_x < \infty$ then $|x(n-k)| \le M_x < \infty$

$$|y(n)| \le \sum_{k=-\infty}^{\infty} |h(k)| M_x$$

$$\le M_x \sum_{k=-\infty}^{\infty} |h(k)|$$

$\therefore$ a LTI system characterised by impulse response $h(n)$ to be BIBO stable,

if $\boxed{\sum_{k=-\infty}^{\infty} |h(k)| < \infty \rightarrow \text{finite}}$ i.e., absolutely summable.

Similarly, for a continuous LTI system characterized by a impulse response $h(t)$ to be BIBO stable if,

$\boxed{\int_{-\infty}^{\infty} h(\tau)\, d\tau < \infty}$ i.e., absolutely integrable.

4) Invertibility :-

A system is invertible only if its input can be recovered from its output by connecting it with a system called inverse system. Thus, if an LTI system is invertible, then it has inverse system.

The unit-step Response of an LTI system in terms of impulse response :-

If the input to the LTI system is unit-step function, then the response of the system is known as step-Response.

Consider a LTI system,

$$y(n) = x(n) * h(n)$$

Since convolution is commutative, hence

$$y(n) = h(n) * x(n)$$

If the input is unit step i.e., $x(n) = u(n)$, then the step response is given by,

$$s(n) = h(n) * u(n)$$

$$= \sum_{k=-\infty}^{\infty} h(k)\, u(n-k)$$

W.K.T $u(n-k) = 1 \; ; \; n-k \geq 0$ (or) $k \leq n$

$= 0 \; ; \; n-k < 0$ (or) $k > n$

$$\therefore \boxed{s(n) = \sum_{k=-\infty}^{n} h(k)}$$

$\therefore$ Step response of a discrete-time LTI system is the running sum of the impulse response.

Similarly, for continuous-time LTI system,

$$y(t) = h(t) * x(t)$$

If the input is unit step i.e., $x(t) = u(t)$, then the step response is given by,

$$s(t) = h(t) * u(t)$$

$$= \int_{-\infty}^{\infty} h(\tau)\, u(t-\tau)\, d\tau$$

W.K.T $u(t-\tau) = 1 \; ; \; t-\tau \geq 0$ (or) $\tau \leq t$

$0 \; ; \; t-\tau < 0$ (or) $\tau > t$

$$\therefore \boxed{s(t) = \int_{-\infty}^{t} h(\tau)\, d\tau}$$

$\therefore$ the step response of a continuous-time LTI system is the running integral of the impulse response.

Note :-

1) $\delta(t)$, $\delta(n)$ is memoryless.

2) In negative time if signal exists, non-causal otherwise causal.

3) For converging, it is stable, for diverging it is unstable.

Property	Continuous-time System	Discrete-time System
Commutative	$x(t) * h(t) = h(t) * x(t)$	$x(n) * h(n) = h(n) * x(n)$
Distributive	$x(t) * \{h_1(t) + h_2(t)\} = x(t) * h_1(t) + x(t) * h_2(t)$	$x(n) * \{h_1(n) + h_2(n)\} = x(n) * h_1(n) + x(n) * h_2(n)$
Associative	$x(t) * \{h_1(t) * h_2(t)\} = \{x(t) * h_1(t)\} * h_2(t)$	$x(n) * \{h_1(n) * h_2(n)\} = \{x(n) * h_1(n)\} * h_2(n)$
Memoryless	$h(t) = 0$ for $t \neq 0$	$h(n) = 0$ for $n \neq 0$
Causality	$h(t) = 0$ for $t < 0$	$h(n) = 0$ for $n < 0$
Stability	$\int_{-\infty}^{\infty} \lvert h(\tau) \rvert d\tau < \infty$	$\sum_{K=-\infty}^{\infty} \lvert h(K) \rvert < \infty$
Invertibility	$h(t) * h_1(t) = \delta(t)$	$h(n) * h_1(n) = \delta(n)$
Step response	$S(t) = \int_{-\infty}^{t} h(\tau) d\tau$	$S(n) = \sum_{K=-\infty}^{n} h(K)$

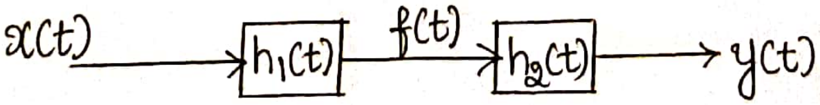
System Interconnection :-

Mathematical operation performed by a system can be represented interms of basic operations such as summer, multiplier and system function.

LTI systems can be connected in 2 different ways :

1. Series Connection
2. parallel Connection.

Series Connection :-



Let the output of the 1st system be $f(t)$

$$f(t) = x(t) * h_1(t)$$

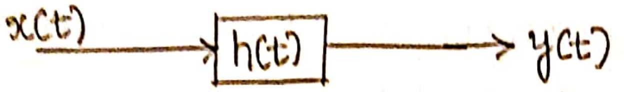
and $y(t) = f(t) * h_2(t)$

$$= [x(t) * h_1(t)] * h_2(t)$$

$y(t) = x(t) * [h_1(t) * h_2(t)]$; by Associative property

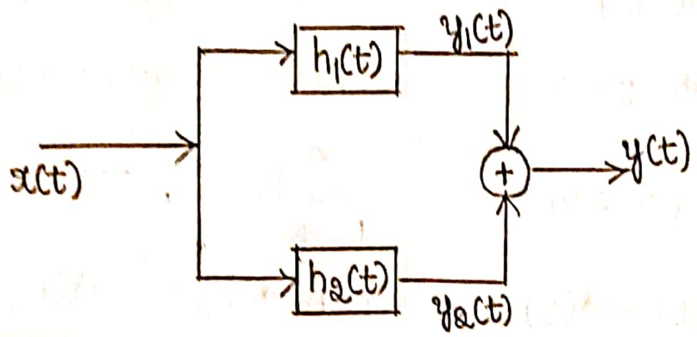
let $h(t) = h_1(t) * h_2(t)$

then $y(t) = x(t) * h(t)$



i.e., overall response of the system is $\boxed{h(t) = h_1(t) * h_2(t)}$

Parallel Connection :—



$$y_1(t) = x(t) * h_1(t)$$

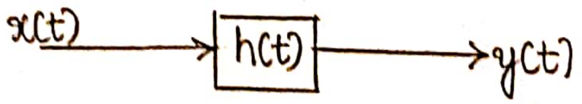
$$y_2(t) = x(t) * h_2(t)$$

$$y(t) = y_1(t) + y_2(t) = x(t) * h_1(t) + x(t) * h_2(t)$$

$$= x(t) * \{h_1(t) + h_2(t)\}$$

Let $h(t) = h_1(t) + h_2(t)$

then $y(t) = x(t) * h(t)$



i.e., overall response of the system is $\boxed{h(t) = h_1(t) + h_2(t)}$

Module-3

Properties of impulse response and System Interconnection.

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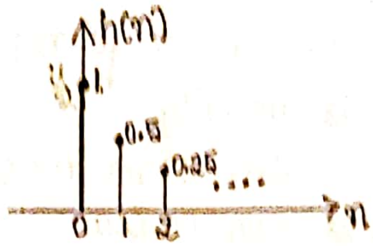
I. The following are the impulse responses of discrete-time LTI systems. Determine, whether each system is (i) memoryless (ii) causal and (iii) stable. Justify your answer.

1) $h(n) = \left(\frac{1}{2}\right)^n u(n)$



Sol^n:-

$$h(n) = \begin{cases} \left(\frac{1}{2}\right)^n & ; n \geq 0 \\ 0 & ; n < 0 \end{cases}$$

(i) Memoryless :- $h(n) = 0$ for $n \neq 0$

$\therefore$ S/m is memory.

(ii) Causal :- $h(n) = 0$ for $n < 0$

$\therefore$ S/m is causal.

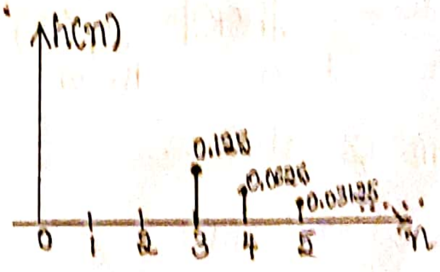
(iii) Stable :- $\sum_{K=-\infty}^{\infty} |h(K)| = \sum_{K=0}^{\infty} \left(\frac{1}{2}\right)^K$

$$= \frac{1}{1-\frac{1}{2}}$$

$$= 2 < \infty$$

$\therefore$ It is absolutely summable. Hence, it is stable.

2) $h(n) = \left(\frac{1}{2}\right)^n u(n-3)$



Sol^n: $h(n) = \begin{cases} \left(\frac{1}{2}\right)^n & ; n \geq 3 \\ 0 & ; n < 3 \end{cases}$

i) Memoryless :- $h(n) = 0$ for $n \neq 0$

$\therefore$ S/m is memory.

ii) Causal :- $h(n) = 0$ for $n < 0$

$\therefore$ S/m is causal.

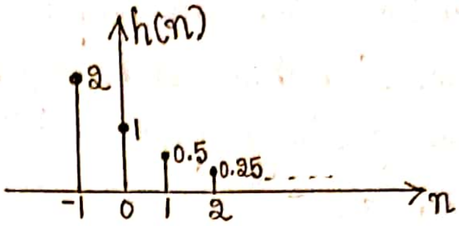
iii) Stable :- $\sum_{K=-\infty}^{\infty}|h(K)| = \sum_{K=3}^{\infty}\left(\frac{1}{2}\right)^K = \frac{\left(\frac{1}{2}\right)^3}{1-\frac{1}{2}} = \frac{1}{4} < \infty$. It is absolutely summable.

$\therefore$ S/m is stable.

3) $h(n) = \left(\frac{1}{2}\right)^n u(n+1)$

Sol^n:-

$$h(n) = \begin{cases} \left(\frac{1}{2}\right)^n & ; n \geq -1 \\ 0 & ; n < -1 \end{cases}$$



i) Memoryless :- $h(n) = 0$ for $n \neq 0$

$\therefore$ S/m is memory.

ii) Causal :- $h(n) = 0$ for $n < 0$

$\therefore$ S/m is non-causal.

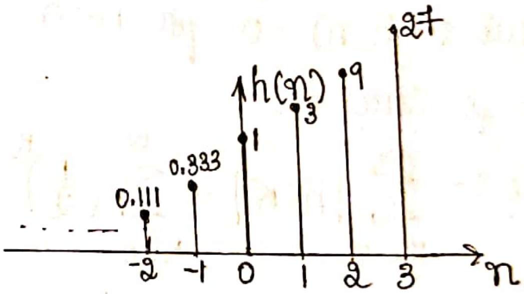
iii) Stable :- $\sum_{K=-\infty}^{\infty} h(K) = \sum_{K=-1}^{\infty}\left(\frac{1}{2}\right)^K = \frac{\left(\frac{1}{2}\right)^{-1}}{1-\frac{1}{2}} = 4 < \infty$. It is absolutely summable.

$\therefore$ S/m is stable.

4) $h(n) = 3^n u(-n+3)$

Sol^n:-

$$h(n) = \begin{cases} 3^n & ; -\infty \leq n \leq 3 \\ 0 & ; n > 3 \end{cases}$$



i) Memoryless :- $h(n) = 0$ for $n \neq 0$

$\therefore$ S/m is memory.

ii) Causal :- $h(n) = 0$ for $n < 0$

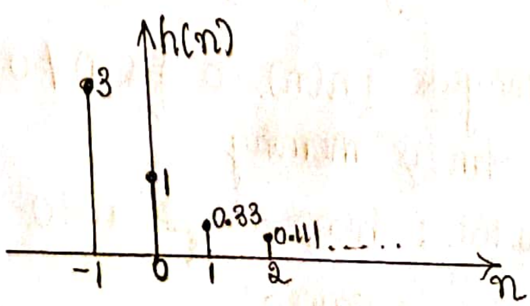
$\therefore$ S/m is non-causal.

iii) Stable :- $\sum_{K=-\infty}^{\infty}|h(K)| = \sum_{K=-\infty}^{3}(3)^n = \sum_{K=-3}^{\infty}\left(\frac{1}{3}\right)^n = \frac{\left(\frac{1}{3}\right)^{-3}}{1-\frac{1}{3}} = \frac{3}{2} \times 27 = \frac{81}{2} < \infty$

$\therefore$ S/m is stable.

5) $h(n) = \left(\frac{1}{3}\right)^n u(n+1)$

Sol^n: $h(n) = \begin{cases} (1/3)^n & ; n \geq -1 \\ 0 & ; n < -1 \end{cases}$



i) Memoryless :- $h(n) = 0$ for $n \neq 0$

$\therefore$ S/m is memory.

Module-4

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Fourier Representation of Aperiodic Signals

Introduction :-

The Convolution sum and integral provides a convenient way to find the response of an LTI system if its impulse response is known and also an LTI system can be completely characterized by its impulse response.

A signal can be represented as a weighted superposition of complex sinusoidals. If such a signal is applied to an LTI system, then the system output is a weighted superposition of the system outputs to each complex sinusoidals.

The representations of signals and systems using complex sinusoidals is called Fourier representation.

Fourier Representations for Signal Classes :-

Depending on the periodic nature of a signal, there are four distinct Fourier representations.

Periodic signals have Fourier series representations, whereas non-periodic signals have Fourier transform representations.

If the signal is periodic continuous-time signal, the representation is termed as Fourier Series (FS) whereas for periodic discrete-time signal, the representation is termed as discrete-time Fourier Series (DTFS).

Similarly, if the signal is non-periodic continuous-time signal, the representation is termed as Fourier transform (CFT) whereas for non-periodic discrete-time signal, the representation is termed as discrete-time Fourier transform (DTFT).

Time properties of a signal and its appropriate Fourier representation:-

Time property.	periodic	Non-periodic
Continuous	Fourier Series (FS)	Fourier transform (FT)
Discrete	Discrete-time Fourier Series (DTFS)	Discrete-time Fourier transform (DTFT)

2) Orthogonality of Complex Sinusoidal Signals :-

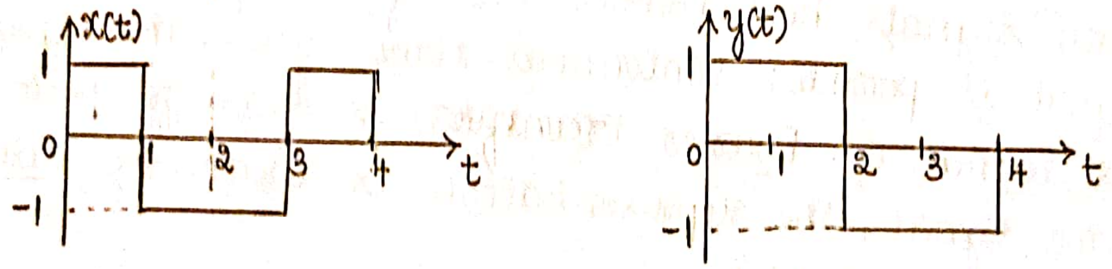
W.K.T Any signal can be expressed as a weighted superposition of complex sinusoidals.

Consider two continuous-time signals $x(t)$ and $y(t)$. These two signals are said to be orthogonal over the interval (a,b) if,

$$\int_a^b x(t)\, y^*(t)\, dt = 0$$

Where, $y^*(t)$ is the complex conjugate of $y(t)$.

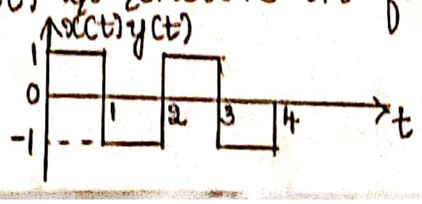
Ex:- Consider two continuous-time signals $x(t)$ and $y(t)$ as shown below,



Let us check for the orthogonality of $x(t)$ and $y(t)$ over the interval $(0,4)$. Since, $y(t)$ is real $y^*(t) = y(t)$. Therefore, to check the orthogonality we have to Evaluate,

$$\int_0^4 x(t)\, y(t)\, dt$$

The signal $x(t)y(t)$ is shown in fig below,



$$\therefore \int_0^4 x(t)\,y(t)\,dt = \int_0^1 1.dt + \int_1^2 (-1)dt + \int_2^3 1.dt + \int_3^4 (-1)\,dt = 0$$

Therefore, $x(t)$ and $y(t)$ are orthogonal over the interval $(0,4)$.

Similarly, two discrete-time signals $x(n)$ and $y(n)$ are said to be orthogonal over the interval (N_1, N_2) if,

$$\sum_{n=N_1}^{N_2} x_k(n)\,y_m^*(n) = A_K \;;\; K = m$$
$$= 0 \;;\; K \neq m.$$

where, 'A_K' is a constant.

3) Fourier Transform :-

Non-periodic signals can be represented with the help of Fourier transform. Fourier transform provides effective reversible link between frequency-domain and time-domain representation of the signal.

For non-periodic signals $T_0 \to \infty$. Hence, $w_0 = 0$. Therefore, spacing between the spectral components becomes infinitesi--mal and hence the spectrum appears to be continuous.

Definition of Fourier transform :-

The fourier transform of $x(t)$ is defined as,

$$X(w) = \int_{-\infty}^{\infty} x(t)\,e^{-jwt}\,dt \quad (\text{or}) \quad X(f) = \int_{-\infty}^{\infty} x(t)\,e^{-j2\pi ft}\,dt \text{ (Analysis Equation)}.$$

Here, '$x(t)$' is time-domain representation of the signal and '$X(w)$' (or) '$X(f)$' is frequency-domain representation of the signal, 'w' is the frequency.

Sometimes, $X(w)$ is also written as $X(jw)$.

Similarly, $x(t)$ can be obtained from $X(w)$ by inverse Fourier transform. i.e.,

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(w)\,e^{jwt}\,dw = \int_{-\infty}^{\infty} X(f)\,e^{j2\pi ft}\,df \text{ (Synthesis Equation)}$$

A Fourier transform pair is represented as,

$$x(t) \xleftrightarrow{FT} X(\omega) \text{ (or) } x(t) \xleftrightarrow{FT} X(f).$$

Existence of Fourier transform - Dirichlet Conditions :-

The Fourier transform $X(\omega)$ for a continuous-time signal $x(t)$ exists if the following conditions (referred to as Dirichlet conditions) are satisfied:

i) Single-Valued property :- $x(t)$ must have only value at any time instant over a finite time interval 'T'.

ii) Finite discontinuities :- $x(t)$ should have at the most finite number of discontinuities over a finite time interval 'T'.

iii) Finite peaks :- The signal $x(t)$ should have finite number of maxima and minima over a finite time interval 'T'.

iv) Absolute integrability :- $x(t)$ should be absolutely integrable

i.e., $\int_{-\infty}^{\infty} |x(t)| dt < \infty$.

The above conditions are sufficient, but not necessary for the signal to be Fourier transformable.

properties of Fourier transform :-

The different properties of Fourier transform are :

i) Linearity
ii) Time shift
iii) Frequency shift
iv) Time Scaling
v) Time differentiation
vi) Frequency differentiation
vii) Integration
viii) Convolution
ix) Modulation

x) parseval's theorem

xi) Duality

xii) Symmetry

i) Linearity :-

If $x(t) \xleftrightarrow{FT} X(j\omega)$ and

$y(t) \xleftrightarrow{FT} Y(j\omega)$

then, $z(t) = a\,x(t) + b\,y(t) \xleftrightarrow{FT} Z(j\omega) = a\,X(j\omega) + b\,Y(j\omega)$

Meaning :-

The Fourier transform of linear combination of the signals is equal to linear combination of their Fourier transforms. It is also called superposition.

proof :- $X(j\omega) = \int_{-\infty}^{\infty} x(t)\,e^{-j\omega t}\,dt$

$$Y(j\omega) = \int_{-\infty}^{\infty} y(t)\,e^{-j\omega t}\,dt$$

$$\therefore Z(j\omega) = \int_{-\infty}^{\infty} z(t)\,e^{-j\omega t}\,dt$$

$$= \int_{-\infty}^{\infty} [a\,x(t) + b\,y(t)]\,e^{-j\omega t}\,dt$$

$$= a\int_{-\infty}^{\infty} x(t)\,e^{-j\omega t}\,dt + b\int_{-\infty}^{\infty} y(t)\,e^{-j\omega t}\,dt$$

$$Z(j\omega) = a\,X(j\omega) + b\,Y(j\omega)$$

Hence, the proof.

ii) Time Shift :-

If $x(t) \xleftrightarrow{FT} X(j\omega)$

then $y(t) = x(t-t_0) \xleftrightarrow{FT} Y(j\omega) = e^{-j\omega t_0}\,X(j\omega)$

Meaning :-

A shift of 't_0' in time-domain is equivalent to introducing a phase shift of $-\omega t_0$. But amplitude remains same.

Proof :-

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$\therefore Y(j\omega) = \int_{-\infty}^{\infty} y(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} x(t-t_0)\, e^{-j\omega t}\, dt$$

put $t-t_0 = a$, then $dt = da$

$$Y(j\omega) = \int_{-\infty}^{\infty} x(a)\, e^{-j\omega(a+t_0)}\, da$$

$$= e^{-j\omega t_0} \int_{-\infty}^{\infty} x(a)\, e^{-j\omega a}\, da$$

$$Y(j\omega) = e^{-j\omega t_0}\, X(j\omega)$$

Hence, the proof.

(iii) Frequency Shift :-

If $x(t) \xleftrightarrow{FT} X(j\omega)$

then $y(t) = e^{j\beta t} x(t) \xleftrightarrow{FT} Y(j\omega) = X(j(\omega-\beta))$

Meaning :-

It states that by shifting the frequency by 'β' in frequency domain is equivalent to multiplying the time-domain signal by $e^{j\beta t}$.

Proof :-

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$\therefore Y(j\omega) = \int_{-\infty}^{\infty} y(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} e^{j\beta t}\, x(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} x(t)\, e^{-j(\omega-\beta)t}\, dt$$

Module-4

Fourier Representation of Aperiodic Signals

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Problems on Fourier transform :-

1) Obtain the Fourier transform of the signal, $x(t) = e^{-at}u(t)$, $a>0$. Draw its magnitude and phase spectra.

Soln:- W.K.T $X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$

$$= \int_{0}^{\infty} e^{-at}\, e^{-j\omega t}\, dt$$

$$= \int_{0}^{\infty} e^{-(a+j\omega)t}\, dt$$

$$= \left. \frac{e^{-(a+j\omega)t}}{-(a+j\omega)} \right|_{0}^{\infty}$$

$$= -\frac{1}{(a+j\omega)}(0-1)$$

$$\left\langle X(\omega) = \frac{1}{(a+j\omega)} \; ; a>0 \right\rangle$$

To obtain magnitude and phase spectrum :-

$$X(\omega) = \frac{1}{(a+j\omega)} = \frac{1}{a+j\omega} \times \frac{a-j\omega}{a-j\omega} = \frac{a-j\omega}{a^2+\omega^2} = \frac{a}{a^2+\omega^2} - j\frac{\omega}{a^2+\omega^2}$$

$$\therefore |X(\omega)| = \sqrt{\left(\frac{a}{a^2+\omega^2}\right)^2 + \left(\frac{\omega}{a^2+b^2}\right)^2} = \sqrt{\frac{a^2+\omega^2}{(a^2+\omega^2)^2}} = \sqrt{\frac{1}{(a^2+\omega^2)}}$$

$$\angle X(\omega) = \tan^{-1}\left[\frac{\frac{-\omega}{a^2+\omega^2}}{\frac{a}{a^2+\omega^2}}\right] = -\tan^{-1}\left(\frac{\omega}{a}\right)$$

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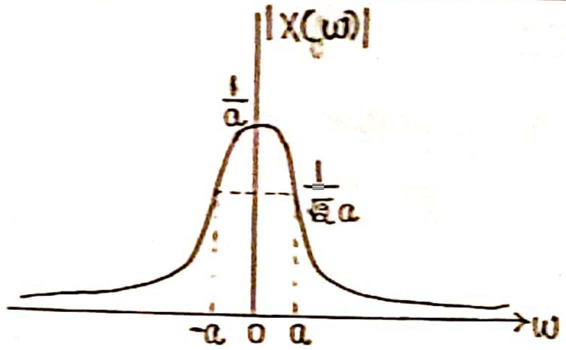
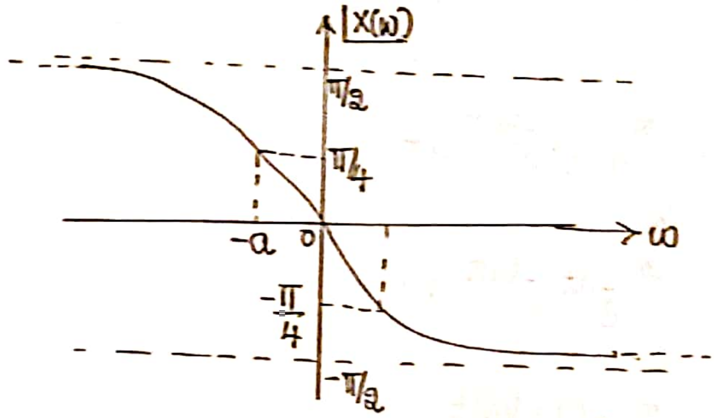




2) Find the Fourier transform of $x(t) = e^{at} u(-t)$. Draw its Spectrum.

Soln:-

W.K.T $X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$

$$= \int_{-\infty}^{0} e^{at}\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{0} e^{(a-j\omega)t}\, dt$$

$$= \left. \frac{e^{(a-j\omega)t}}{(a-j\omega)} \right|_{-\infty}^{0}$$

$$= \frac{1}{(a-j\omega)} \left[1 - 0 \right]$$

$$X(\omega) = \frac{1}{(a-j\omega)}$$

To obtain magnitude and phase Spectra :-

$$X(\omega) = \frac{1}{(a-j\omega)} \times \frac{(a+j\omega)}{(a+j\omega)} = \frac{a+j\omega}{a^2+\omega^2} = \frac{a}{a^2+\omega^2} + j\frac{\omega}{a^2+\omega^2}$$

$$X(\omega) = \left.\frac{e^{(a-j\omega)t}}{(a-j\omega)}\right|_{-\infty}^{0} + \left.\frac{e^{-(a+j\omega)t}}{-(a+j\omega)}\right|_{0}^{\infty}$$

$$= \frac{1}{(a-j\omega)}\{1-0\} - \frac{1}{(a+j\omega)}\{0-1\}$$

$$= \frac{1}{(a-j\omega)} + \frac{1}{(a+j\omega)}$$

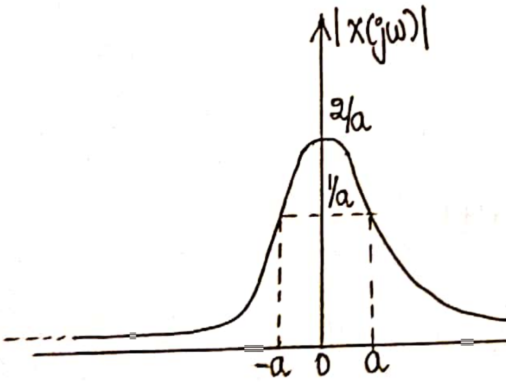
$$= \frac{a+j\omega + a-j\omega}{a^2+\omega^2}$$

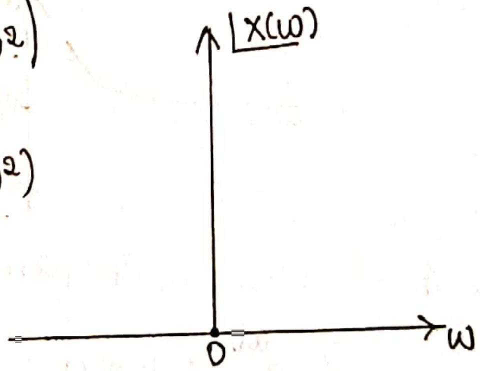
$$\left\langle X(\omega) = \frac{2a}{a^2+\omega^2} \right\rangle$$

Magnitude and phase Spectrum:-

Since, the given $x(t)$ is Even symmetric, $X(j\omega)$ is purely real.

$$|X(j\omega)| = \sqrt{\left(\frac{2a}{a^2+\omega^2}\right)^2} = \frac{2a}{(a^2+\omega^2)}$$





4) Determine the fourier transform of the unit impulse function. i.e., $x(t) = \delta(t)$.

Soln:-

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} \delta(t)\, e^{-j\omega t}\, dt$$

$$= \left. e^{-j\omega t}\right|_{t=0} \quad [\because \text{Sifting property}]$$

$$\langle X(\omega) = 1 \rangle \qquad |X(\omega)| = \sqrt{1^2} = 1 \; ; \; \angle X(\omega) = \tan^{-1}\left(\frac{0}{1}\right) = 0^\circ$$

Module-4

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Impulse Sampling and Reconstruction

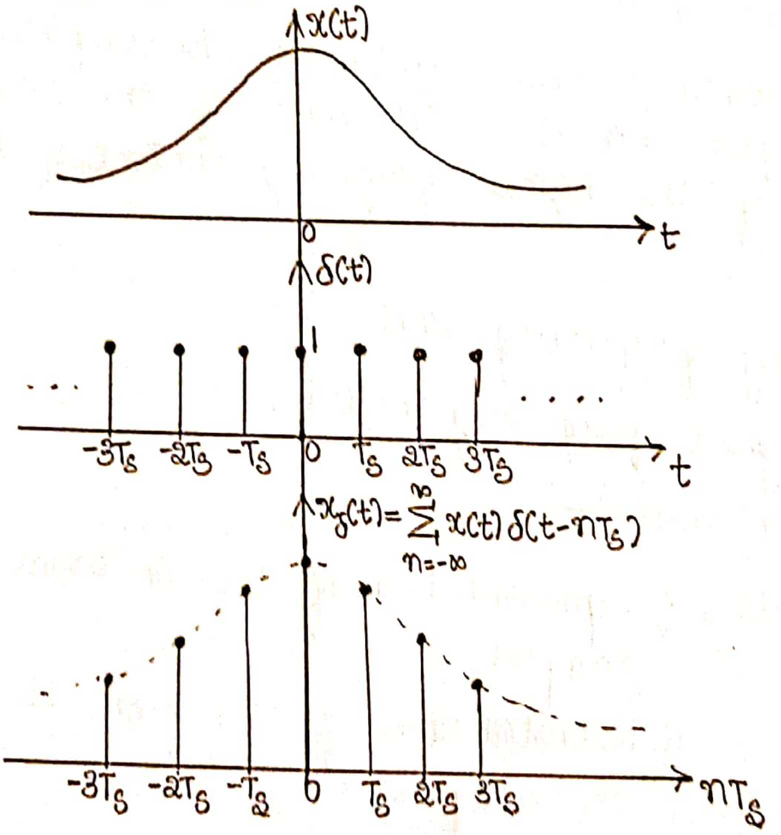
i) Representation of Continuous-time Signals by its samples :-

Continuous-time signals are represented by its samples for two reasons :

i) Continuous-time signal cannot be processed in the digital processor (or) Computer.

ii) To enable digital transmission of continuous-time signals.

Fig. below shows the continuous-time signal and its sampled discrete-time signal. In the fig, continuous-time signal is sampled at $t = 0, T_s, 2T_s, 3T_s, \ldots$ and so on.



fig(a): CT and its DT signal

Here sampling theorem gives the criteria for spacing 'T_s' between two successive samples. The samples $x_s(t)$ must represent all the information contained in $x(t)$. The sampled signal $x_s(t)$ is called discrete-time (DT) signal. It is analyzed with the help of DTFT and Z-transform.

ii) Sampling theorem for low-pass (LP) Signals :-

A low-pass (or) LP signal contains frequencies from 1Hz to some higher value.

1) A band limited signal of finite energy, which has no frequency components higher than 'W' Hertz, is completely described by specifying the values of the signal at instants of time separated by $\frac{1}{2W}$ seconds and

2) A bandlimited signal of finite energy, which has no frequency components higher than 'W' Hertz, may be completely recovered from the knowledge of its samples taken at the rate of 2W samples per second.

The first part of above statement tells about sampling of the signal and second part tells about reconstruction of the signal.

Statement :- A continuous time signal can be completely represented in its samples and recovered back if the sampling frequency is twice of the highest frequency content of the signal.

i.e., $f_s \geq 2W$

Here, 'f_s' is sampling frequency and

'W' is the highest frequency content.

Proof of Sampling theorem :-

There are two parts : i) Representation of $x(t)$ in terms of its samples.

ii) Reconstruction of $x(t)$ from its samples.

i) Representation of $x(t)$ in its samples $x(nT_s)$:-

Step 1 :- Define $x_\delta(t)$

From the fig, the sampled signal $x_\delta(t)$ is given by,

$$x_\delta(t) = \sum_{n=-\infty}^{\infty} x(t)\,\delta(t-nT_s) \longrightarrow (1)$$

where $x_\delta(t)$ is the product of $x(t)$ and impulse train $\delta(t)$.

In the above Eq^n, $\delta(t-nT_s)$ indicates the samples placed at $\pm T_s, \pm 2T_s, \pm 3T_s$ and so on.

Step 2 :- Fourier transform of $x_\delta(t)$ i.e., $X_\delta(f)$

Taking FT of Eq(1),

$$X_\delta(f) = FT\left\{\sum_{n=-\infty}^{\infty} x(t)\,\delta(t-nT_s)\right\} = FT\{\text{product of } x(t) \text{ and impulse train}\}$$

W.K.T FT of product in time domain becomes convolution in frequency i.e.,

$$X_\delta(f) = FT\{x(t)\} * FT\{\delta(t-nT_s)\} \longrightarrow (2)$$

By definition, $x(t) \xleftrightarrow{FT} X(f)$ and

$$\delta(t-nT_s) \xleftrightarrow{FT} f_s \sum_{n=-\infty}^{\infty} \delta(f-nT_s)$$

$\therefore$ Eq(2) becomes,

$$X_\delta(f) = X(f) * f_s \sum_{n=-\infty}^{\infty} \delta(f-nf_s)$$

Since convolution is linear,

$$X_\delta(f) = f_s \sum_{n=-\infty}^{\infty} X(f) * \delta(f-nf_s)$$

$$= f_s \sum_{n=-\infty}^{\infty} X(f-nf_s) \qquad \text{[By shifting property of impulse function]}$$

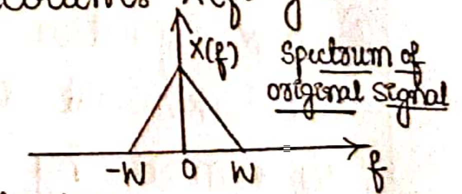
$$= \ldots + f_s X(f+2f_s) + f_s X(f+f_s) + f_s X(f) + f_s X(f-f_s) + f_s X(f-2f_s) + \ldots$$

Comments :-

i) The RHS of above equation shows that $X(f)$ is placed at $\pm f_s, \pm 2f_s, \pm 3f_s, \ldots$

ii) This means $X(f)$ is periodic in f_s.

iii) If sampling frequency is $f_s = 2W$, then the spectrums $X(f)$ just touch each other.

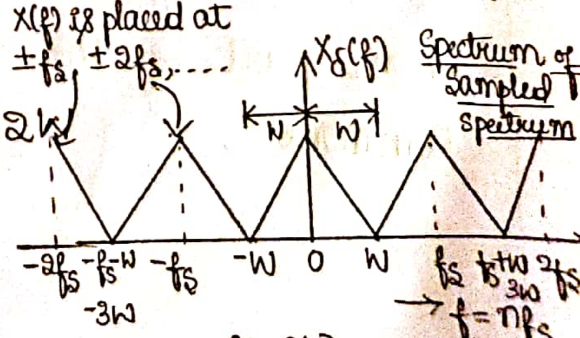


Step 3 : Relation between $X(f)$ and $X_\delta(f)$.

Let us assume that $f_s = 2W$, then

$$X_\delta(f) = f_s X(f) \quad \text{for } -W \le f \le W \text{ and } f_s = 2W$$

(or) $$X(f) = \frac{1}{f_s} X_\delta(f) \longrightarrow (3)$$



fig(b)

Step 4 : Relation between $x(t)$ and $x(nT_s)$

DTFT is, $X(\Omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\Omega n}$

$$\therefore X(f) = \sum_{n=-\infty}^{\infty} x(n) e^{-j2\pi f n} \longrightarrow (4)$$

In above Equation, 'f' is the frequency of discrete-time signal. If we replace $X(f)$ by $X_\delta(f)$, then 'f' becomes frequency of continuous-time signal.

i.e., $X_\delta(f) = \sum_{n=-\infty}^{\infty} x(n) e^{-j2\pi \frac{f}{f_s} n}$

In above Equation, 'f' is frequency of continuous-time signal and $\frac{f}{f_s}$ = frequency of discrete-time signal in Eq(4).

Since $x(n) = x(nT_s)$, i.e., samples of $x(t)$, then

$$X_\delta(f) = \sum_{n=-\infty}^{\infty} x(nT_s) e^{-j2\pi f n T_s} \quad \text{since } \frac{1}{f_s} = T_s$$

Putting above expression in Eq(3),

$$X(f) = \frac{1}{f_s} \sum_{n=-\infty}^{\infty} x(nT_s) e^{-j2\pi f n T_s}$$

Inverse fourier transform (IFT) of above Equation gives $x(t)$ i.e.,

$$x(t) = \text{IFT}\left\{ \frac{1}{f_s} \sum_{n=-\infty}^{\infty} x(nT_s) e^{-j2\pi f n T_s} \right\} \longrightarrow (5)$$

Comments :—

i) Here $x(t)$ is represented completely in terms of $x(nT_s)$.

ii) Above Equation holds for $f_s = 2W$. This means if the samples are taken at the rate of $2W$ (or) higher, $x(t)$ is completely represented by its samples.

iii) First part of the sampling theorem is proved by above two comments.

ii) Reconstruction of x(t) from its samples :-

Step 1 :- Take inverse Fourier transform of X(f) which is in terms of $X_\delta(f)$

The IFT of Eq(5) becomes,

$$x(t) = \int_{-\infty}^{\infty} \left\{ \frac{1}{f_s} \sum_{n=-\infty}^{\infty} x(nT_s) e^{-j2\pi f n T_s} \right\} e^{j2\pi f t} \, df$$

Here the integration can be taken from $-W \le f \le W$. Since $X(f) = \frac{1}{f_s} X_\delta(f)$ for $-W \le f \le W$

$$\therefore x(t) = \int_{-W}^{W} \frac{1}{f_s} \sum_{n=-\infty}^{\infty} x(nT_s) e^{-j2\pi f n T_s} \cdot e^{j2\pi f t} \, dt$$

Interchanging the order of summation and integration,

$$x(t) = \sum_{n=-\infty}^{\infty} x(nT_s) \times \frac{1}{f_s} \int_{-W}^{W} e^{j2\pi f (t-nT_s)} \, df$$

$$= \sum_{n=-\infty}^{\infty} x(nT_s) \times \frac{1}{f_s} \times \left[\frac{e^{j(2\pi f(t-nT_s))}}{j2\pi(t-nT_s)} \right]_{-W}^{W}$$

$$= \sum_{n=-\infty}^{\infty} x(nT_s) \times \frac{1}{f_s} \left\{ \frac{e^{j2\pi W(t-nT_s)} - e^{-j2\pi W(t-nT_s)}}{j2\pi(t-nT_s)} \right\}$$

$$= \sum_{n=-\infty}^{\infty} x(nT_s) \times \frac{1}{f_s} \times \frac{\sin 2\pi W(t-nT_s)}{\pi(t-nT_s)}$$

$$= \sum_{n=-\infty}^{\infty} x(nT_s) \times \frac{\sin \pi(2Wt - 2WnT_s)}{\pi(f_s t - f_s n T_s)}$$

Here $f_s = 2W$, hence $T_s = \frac{1}{f_s} = \frac{1}{2W}$

$$\therefore x(t) = \sum_{n=-\infty}^{\infty} x(nT_s) \times \frac{\sin \pi\left(2Wt - \cancel{2W} n \times \frac{1}{\cancel{2W}}\right)}{\pi\left(2Wt - \cancel{2W} \times n \times \frac{1}{\cancel{2W}}\right)}$$

iv) Nyquist rate and Nyquist interval :-

Nyquist rate :-

When the sampling rate becomes exactly equal to '2W' samples/sec, for a given bandwidth of 'W' Hertz, then it is called Nyquist rate.

⟨Nyquist rate = 2W Hz⟩

Nyquist interval :-

It is the time interval between any two adjacent samples when sampling rate is Nyquist rate.

$$\left\langle \text{Nyquist interval} = \frac{1}{2W} \text{ seconds} \right\rangle$$

) Reconstruction filter (Interpolation filter)

The reconstructed signal is the succession of sinc pulses weighted by $x(nT_s)$. These pulses are interpolated with the help of a lowpass filter. It is also called reconstruction filter (or) interpolation filter.

Ideal filter :-

fig below shows the spectrum of sampled signal and frequency response of required filter. When the sampling frequency is exactly 2W, then the spectrums just touch each other as shown in fig. The spectrum of original signal, X(f) can be filtered by an ideal filter having passband from $-W \le f \le W$.

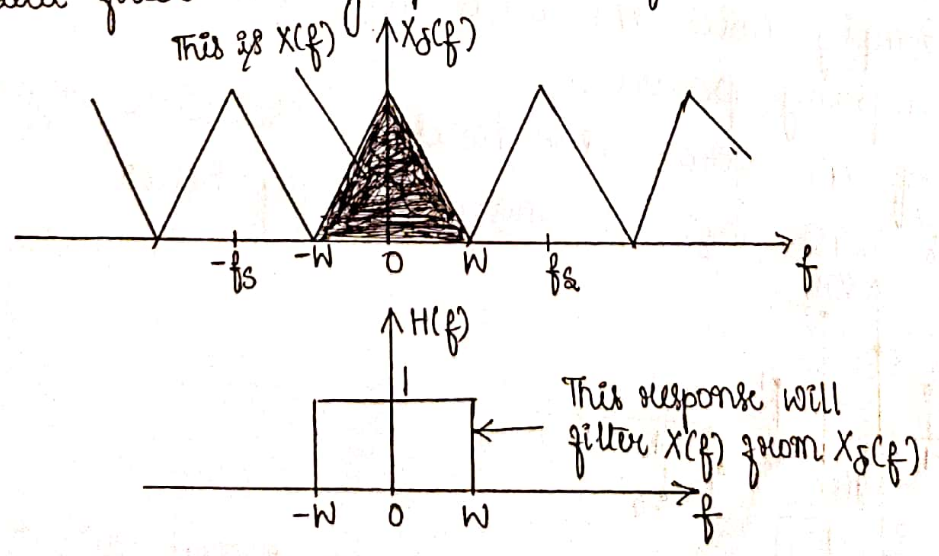


fig : Ideal reconstruction filter

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1) A Signal $x(t) = \cos(5\pi t) + 0.5\cos(10\pi t)$ is ideally sampled with sampling period T_s. Find the minimum sampling frequency (i.e., Nyquist rate).

Soln:- Given $x(t) = \cos(5\pi t) + 0.5\cos(10\pi t) \longrightarrow (1)$

$$\omega_1 = 5\pi \text{ rad/sec} ; \quad f_1 = 2.5\text{Hz}$$
$$\omega_2 = 10\pi \text{ rad/sec} ; \quad f_2 = 5\text{ Hz} = f_{max}$$

highest frequency $W = f_2 = 5\text{Hz}$.

Nyquist rate $= 2W = 2\times 5 = \underline{\underline{10\text{Hz}}}$

2) Specify the Nyquist rate for each of the following signals:
i) $x_1(t) = \text{Sinc}(200t)$ ii) $x_2(t) = \text{Sinc}^2(200t)$

Soln:-

i) Given $x_1(t) = \text{Sinc}(200t)$

$$= \frac{\sin(200\pi t)}{(200\pi t)} \qquad \because \text{Sinc}(\theta) = \frac{\sin(\pi\theta)}{(\pi\theta)}$$

$W = f = 100\text{Hz}$

Nyquist rate $= 2W = 2\times 100 = \underline{\underline{200\text{Hz}}}$

ii) Given $x_2(t) = \text{Sinc}^2(200t)$

$$\sin^2\theta = \frac{1-\cos 2\theta}{2}$$

$$= \left[\frac{\sin(200\pi t)}{200\pi t}\right]^2$$

$$= \frac{1}{(200\pi t)^2}\left[\frac{1}{2}(1-\cos 400\pi t)\right]$$

$W = 200\text{Hz}$

Nyquist rate $= 2W = \underline{\underline{400\text{Hz}}}$

3) Determine the Nyquist rate Corresponding to the following signals.

i) $x(t) = \cos(150\pi t)\sin(100\pi t)$

Solⁿ:-

$$\cos A \sin B = \frac{1}{2}[\sin(A+B) - \sin(A-B)]$$

Given $x(t) = \cos(150\pi t)\sin(100\pi t)$

$$= \frac{1}{2}[\sin(250\pi t) - \sin(50\pi t)]$$

$\Rightarrow f_1 = 125\text{ Hz},\ f_2 = 25\text{ Hz}$

Highest frequency, $W = f_1 = 125\text{ Hz}$.

$\therefore$ Nyquist rate $= 2W = 2\times125 = \underline{250\text{ Hz}}$

ii) $x(t) = \cos^3(200\pi t)$

Solⁿ:-

$$\because \cos^3(\theta) = \frac{3}{4}\cos\theta + \frac{1}{4}\cos(3\theta)$$

$x(t) = \cos^3(200\pi t)$

$$= \frac{3}{4}\cos(200\pi t) + \frac{1}{4}\cos(600\pi t)$$

$f_1 = 100\text{ Hz},\ f_2 = 300\text{ Hz}$

Highest frequency, $W = f_2 = 300\text{ Hz}$

$\therefore$ Nyquist rate $= 2W = 2\times300 = \underline{600\text{ Hz}}$

iii) $x(t) = \text{Sinc}(200t) + \text{Sinc}^2(200t)$

Solⁿ:-

Given $x(t) = \text{Sinc}(200t) + \text{Sinc}^2(200t)$

$$= \frac{\sin(200\pi t)}{(200\pi t)} + \left[\frac{\sin(200\pi t)}{(200\pi t)}\right]^2$$

$$= \frac{\sin(200\pi t)}{(200\pi t)} + \frac{1}{(200\pi t)^2}\left[\frac{1-\cos(400\pi t)}{2}\right]$$

$$x(t) = \frac{\sin(200\pi t)}{(200\pi t)} + \frac{1}{2(200\pi t)^2} - \frac{1}{2(200\pi t)^2}\cos(400\pi t)$$

$f_1 = 100\text{ Hz},\ f_2 = 200\text{ Hz}$

Highest frequency $W = f_2 = 200\text{ Hz}$

$\therefore$ Nyquist rate $= 2W = 2\times200 = \underline{400\text{ Hz}}$

Module-5

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Z-Transforms

Introduction :-

The Z-transform is an extension of the Discrete-time Fourier transform. This extension can be applied to a wider class of signals than the DTFT because there are many signals for which the DTFT doesnot converge but the Z-transform does.

The Z-transform is the discrete-time counterpart to the Laplace transform.

The DTFT can be applied only to stable systems. But Z-transform can be calculated for unstable systems as well.

The solution of linear difference equation becomes easy with the help of Z-transform. The linear difference equation is converted to algebraic equation with the help of Z-transform.

Applications :-

i) Analysis of discrete-time signals and systems.

ii) Digital filter design.

iii) Digital filter / systems synthesis.

Input / output :-

For any input sequence, the Z-transform is complex. It has real and imaginary parts.

$$x(n) \longrightarrow \boxed{\text{Z-transform}} \longrightarrow X(z)$$

Real and imaginary parts

I. The Z-transform :-

Consider, a Complex Exponential input $x(n) = z^n$ to a discrete-time LTI System with impulse response $h(n)$. The System output is given by,

$$y(n) = h(n) * x(n) = \sum_{K=-\infty}^{\infty} h(K)\, x(n-K)$$

$$= \sum_{K=-\infty}^{\infty} h(K)\, z^{n-K}$$

$$= z^n \sum_{K=-\infty}^{\infty} h(K)\, z^{-K}$$

$$= H(z)\, z^n \text{ , where } H(z) = \sum_{K=-\infty}^{\infty} h(K)\, z^{-K}$$

(or) $$H(z) = \sum_{n=-\infty}^{\infty} h(n)\, z^{-n}$$

Where, $H(z)$ is Known as Z-transform of $h(n)$.

Generally, the Z-transform of a discrete-time signal $x(n)$ is given by,

$$Z\{x(n)\} = X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n} \longrightarrow (1)$$

Where, 'z' is a complex variable and is given by,

$$z = re^{j\Omega} = r\cos(\Omega) + jr\sin(\Omega) = Re\{z\} + j\,Img\{z\}$$

Where 'r' is magnitude and 'Ω' is the angle of 'z' respectively.

Substituting $z = re^{j\Omega}$ in Eq(1),

$$X(re^{j\Omega}) = \sum_{n=-\infty}^{\infty} x(n)\,(re^{j\Omega})^{-n}$$

$$X(re^{j\Omega}) = \sum_{n=-\infty}^{\infty} \{x(n)\, r^{-n}\}\, e^{-j\Omega n} \longrightarrow (2)$$

From Eq(2), $X(re^{j\Omega})$ is the discrete-time Fourier transform of the sequence $x(n)\, r^{-n}$.

i.e., $$X(re^{j\Omega}) = F\{x(n)\, r^{-n}\} \longrightarrow (3)$$

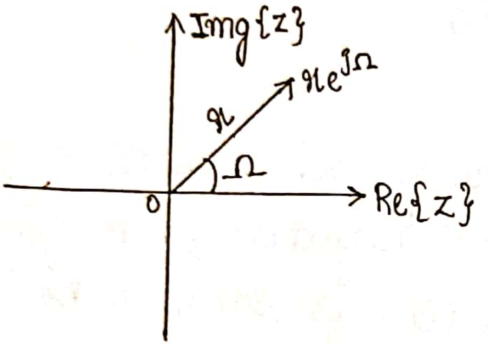
In Eq(3), the Exponential weighting factor 'r^{-n}' may be decaying (or) growing with increasing 'n', depending on whether 'r' is greater than (or) less than unity.

When $r=1$, i.e., $|z|=1$

∴ Eq(2) Becomes,

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\Omega n}$$

i.e., $X(e^{j\Omega}) = X(z)\Big|_{z=e^{j\Omega}}$



Z-plane

Fig shows the Z-plane. A point $Z = re^{j\Omega}$ is located at a distance 'r' from the origin and angle 'Ω' relative to the real axis.

The relationship between x(n) and X(z) with the notation,

$$x(n) \xleftrightarrow{Z} X(z)$$

Z-transform of a discrete-time sequence x(n) is given by,

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n} \longrightarrow (4)$$

$$X(z) = \sum_{n=-\infty}^{\infty} \{x(n)\, r^{-n}\}\, e^{-j\Omega n} \longrightarrow (5)$$

Eq(5) is an infinite sum. For the Existence of X(z) this summation must converge. i.e., the summation should be absolute.

$$\therefore \sum_{n=-\infty}^{\infty} |x(n)\, r^{-n}| < \infty$$

"The Range of Values of 'z' for which X(z) is absolutely summable" i.e., $0<|z|<\infty$

(or)
The range of values of 'r' (radius of the circle) in which $X(z)$ is absolutely summable.

$$r_{min} < |z| < r_{max}.$$

II. Z-transform and ROC of finite duration sequences :-

The range of values of the complex variable z, for which the z-transform converges is called the region of convergence (ROC).

Let us find the z-transform of the following finite-duration sequences and the associated ROC.

I) Right-Sided Sequence :-

A right-sided sequence is one for which $x(n)=0$ for all $n<n_0$, where n_0 is positive (or) negative, but finite. If $n_0 \geq 0$, the resulting sequence, $x(n)$ is said to be either a causal sequence (or) a positive time sequence.

For a causal finite sequence, the ROC is the entire z-plane except for $z=0$.

Ex :- Consider the sequence, $x(n) = \{\underset{\uparrow}{1}, 2, 2, 1\}$

The z-transform of $x(n)$ is,

$$X(z) = \sum_{n=0}^{3} x(n) z^{-n}$$

$$= x(0) z^{-0} + x(1) z^{-1} + x(2) z^{-2} + x(3) z^{-3}$$

$$X(z) = 1 + 2z^{-1} + 2z^{-2} + z^{-3} \; ; \; \text{ROC} : |z| > 0$$

From the above expression for $X(z)$, we find that $X(z)$ is finite for all values of z, except $z=0$. Hence, the ROC is the entire z-plane except at $z=0$.

i.e., ROC : $|z| > 0$.

2) Left-Sided Sequence :—

A Left-Sided Sequence $x(n)$ is one for which $x(n)=0$ for all $n>n_0$, where n_0 is positive (or) negative, but finite. If $n_0 \leq 0$, the resulting sequence $x(n)$ is called anticausal. For such type of finite sequences, the ROC is the entire z-plane excluding $z=\infty$.

Ex:- Consider the sequence, $x(n) = \{1, 1, 2, \underset{\uparrow}{2}\}$

The z-transform of $x(n)$ is

$$X(z) = \sum_{n=-3}^{0} x(n)\, z^{-n} = x(-3)\,z^{3} + x(-2)\,z^{2} + x(-1)\,z^{1} + x(0)\,z^{0}$$

$$X(z) = z^3 + z^2 + 2z + 2 \; ; \; \text{ROC} \; ; \; |z| < \infty$$

The above expression for $X(z)$ becomes infinity at $z=\infty$. Hence, the ROC is the entire z-plane except $z=\infty$. This is explained mathematically by writing the ROC as $|z|<\infty$.

3) Double-Sided Sequence :—

A signal that has finite duration in both the left and right sides is known as double-sided sequence. In this case, the ROC is the entire z-plane except at $z=0$ and $z=\infty$.

Ex:- Consider the sequence $x(n) = \{2, 1, \underset{\uparrow}{1}, 2\}$

The z-transform of $x(n)$ is

$$X(z) = \sum_{n=-2}^{1} x(n)\, z^{-n}$$

$$= x(-2)\,z^{2} + x(-1)\,z^{1} + x(0)\,z^{0} + x(1)\,z^{-1}$$

$$X(z) = 2z^2 + z^1 + 1 + 2z^{-1} \; ; \; \text{ROC} : 0 < |z| < \infty$$

The above expression of $X(z)$ becomes infinity at $z=0$ and $z=\infty$. Hence, the ROC is the entire z-plane except at $z=0$

2) Negative time Exponential Sequence :-

A negative time exponential sequence is defined by,

$x(n) = -b^n u(-n-1)$.

The z-transform of $x(n)$ is,

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{-1} -b^n z^{-n}$$

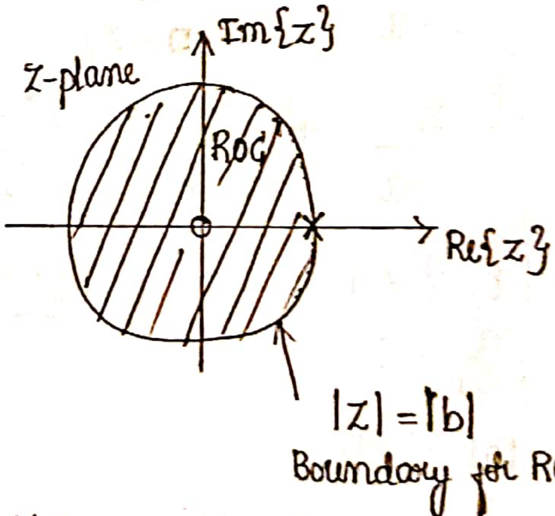
$$= -\sum_{n=-\infty}^{-1} (b z^{-1})^n$$

$$= -\sum_{n=-\infty}^{-1} (b z^{-1})^n$$

$$= -\sum_{n=1}^{\infty} (b z^{-1})^{-n}$$

$$= -\sum_{n=1}^{\infty} (b^{-1} z)^n = -\left[\frac{(b^{-1}z)^1}{1-b^{-1}z}\right] = -\left[\frac{b^{-1}z}{-b^{-1}(z-b)}\right]$$

$$\left\langle X(z) = \frac{z}{z-b} \right\rangle$$



RoC : $|b^{-1} z| < 1$

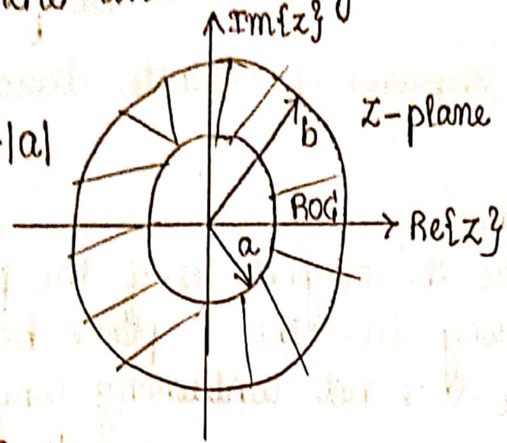
$\left|\frac{z}{b}\right| < 1$

$|z| < |b|$

The RoC and pole-zero plot for this negative time sequence are shown in fig. At this junction, it is important to point out that the positive time exponential has a z-transform with RoC exterior to the circle, $|z| = |b|$, while the negative time exponential sequence has a z-transform with RoC interior and negative time sequences with ROC equal to the intersection of the two regions of convergence.

$$X(z) = \frac{z}{z-a} + \frac{z}{z-b}, \quad (|z|>|a|) \cap (|z|<|b|)$$

If $|b|<|a|$, the above intersection is the null set i.e., the transform does not converge and if $|b|>|a|$, the transform converges in the annular region as shown in fig.



IV. Properties of ROC :-

1) The ROC does not contain any poles.

2) The ROC of X(z) consists of a ring in the z-plane centered about the origin.

3) If $x(n)$ is a finite causal sequence, then the ROC is the entire z-plane except at $z=0$.

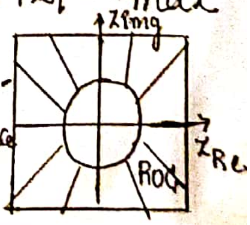
4) If $x(n)$ is a finite non-causal sequence, then the ROC is the entire z-plane except at $z=\infty$.

5) If $x(n)$ is a finite double-sided sequence, then the ROC is the entire z-plane except at $z=0$ and $z=\infty$.

6) If $x(n)$ is a causal infinite length sequence, then, the ROC is of the form $|z|>r_{max}$.

where, r_{max} equals the largest magnitude of any of the poles of X(z). Thus, the ROC is exterior to the circle $|z|=r_{max}$ in the z-plane with the possible exception of $z=\infty$.

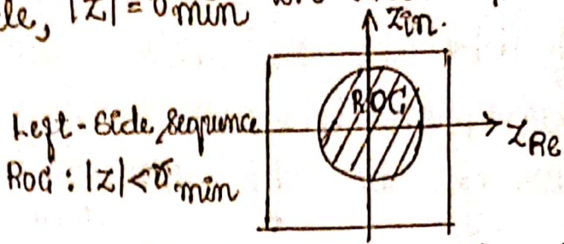
Right-sided sequence
ROC : $|z|>r_{max}$



7) If $x(n)$ is a non-causal infinite length sequence, then the ROC is of the form

$$|z| < r_{min}$$

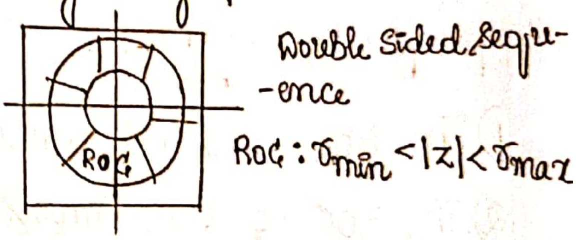
where r_{min} is the smallest magnitude of any of the poles of $X(z)$. Thus, the ROC is interior to the circle, $|z| = r_{min}$ in the z-plane with the possible exception of $z = 0$.



8) If $x(n)$ is a two-sided sequence of infinite-duration, then the ROC is of the form

$$r_{min} < |z| < r_{max}.$$

where r_{min} and r_{max} are the magnitudes of the poles of $X(z)$. Thus, the ROC is an annular ring in the z-plane between the circles $|z| = r_{min}$ and $|z| = r_{max}$ does not containing any poles.



9) The ROC of an LTI stable system contains the unit circle in the z-plane.

10) The ROC must be a connected region.

Module-5

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Z-Transforms

Problems on finite-length sequence :-

I. Determine the Z-transform for the following sequences and indicate its ROC.

1) $x(n) = -2\delta(n) - 3\delta(n+1) + \frac{1}{2}\delta(n-3)$

Sol^n:- $x(n) = \{\underset{\uparrow}{-2}, -3, 0, \frac{1}{2}\}$

W.K.T $X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$

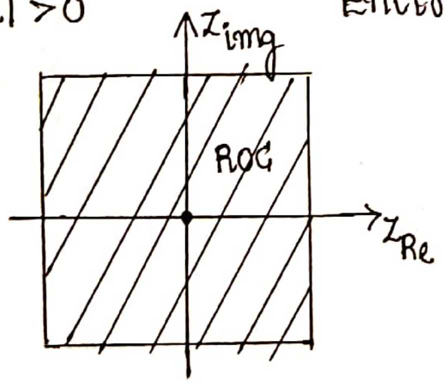
$$= \sum_{n=0}^{3} x(n) z^{-n}$$

$$= x(0) z^{-0} + x(1) z^{-1} + x(2) z^{-2} + x(3) z^{-3}$$

$$X(z) = -2 - 3z^{-1} + \frac{1}{2} z^{-3}$$

$$X(z) = -2 - \frac{3}{z} + \frac{1}{2z^3}$$

ROC : $|z| > 0$ Entire z-plane except $z = 0$.



Note :- For all right-sided sequences, ROC is entire z-plane except $z=0$.

2) $x(n) = \delta(n) + \frac{1}{2}\delta(n-1) + 5\delta(n-3)$

Sol^n: $x(n) = \{\underset{\uparrow}{1}, \frac{1}{2}, 0, 5\}$

W.K.T

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

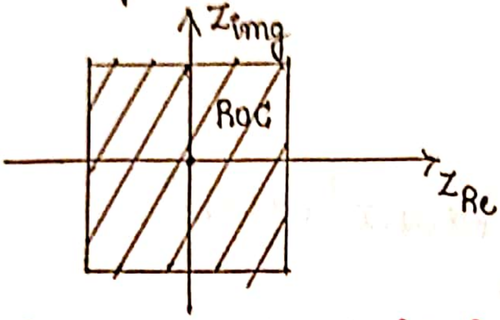
$$= \sum_{n=0}^{3} x(n) z^{-n}$$

$$= x(0) z^{0} + x(1) z^{-1} + x(2) z^{-2} + x(3) z^{-3}$$

$$= 1 + \frac{1}{2} z^{-1} + 5 z^{-3}$$

$$\left\langle X(z) = 1 + \frac{1}{2z} + \frac{5}{z^3} \right\rangle$$

ROC : Entire z-plane Except $z=0$; $|z|>0$



3) $x(n) = \delta(n+2) - 2\delta(n+1) + \frac{1}{2}\delta(n)$

Sol^n:- $x(n) = \{-1, -2, \underset{\uparrow}{\frac{1}{2}}\}$

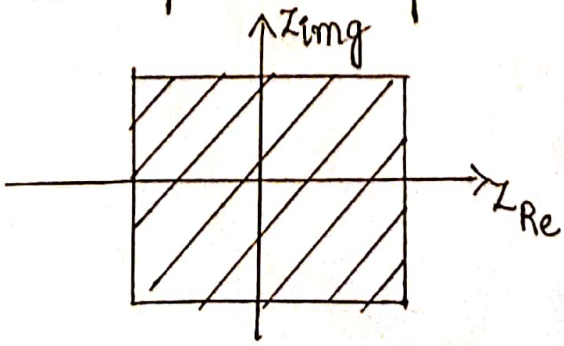
$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=-2}^{0} x(n) z^{-n}$$

$$= x(-2) z^{2} + x(1) z^{1} + x(0) z^{0}$$

$$\left\langle X(z) = -z^{2} - 2z^{1} + \frac{1}{2} \right\rangle$$

ROC : Entire z-plane Except $z=\infty$ (or) $|z| < \infty$



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4) $x(n) = \begin{cases} n+1 & ; -4 \le n \le -2 \\ 1/2 & ; -1 \le n \le 0 \\ 0 & ; \text{Else} \end{cases}$

Sol^n :- $x(n) = \{-3, -2, -1, 1/2, \underset{\uparrow}{1/2}\}$

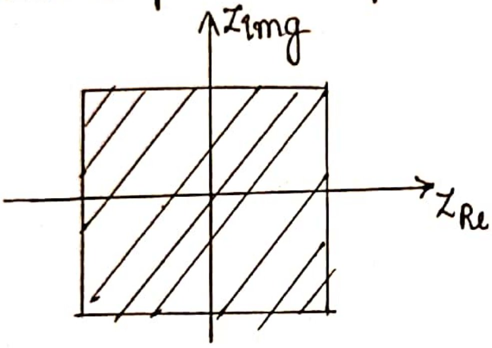
$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=-4}^{0} x(n) z^{-n}$$

$$= x(-4) z^{+4} + x(-3) z^{3} + x(-2) z^{2} + x(-1) z^{1} + x(0) z^{0}$$

$$\left\langle X(z) = -3z^4 - 2z^3 - z^2 + \frac{1}{2} z + \frac{1}{2} \right\rangle$$

ROC : Entire Z-plane Except $Z = \infty$



5) $x(n) = \delta(n)$

Sol^n :- $X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$

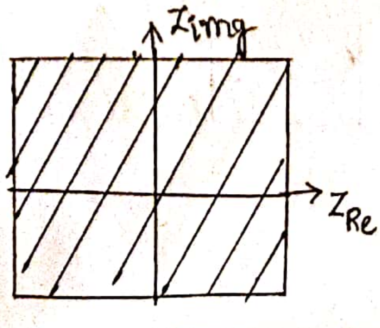
$$= \sum_{n=-\infty}^{\infty} \delta(n) z^{-n}$$

$$= z^{-n} \Big|_{n=0}$$

$$= z^{-0}$$

$$\langle X(z) = 1 \rangle$$

ROC : Entire Z-plane ($\because$ no poles)



Type-I

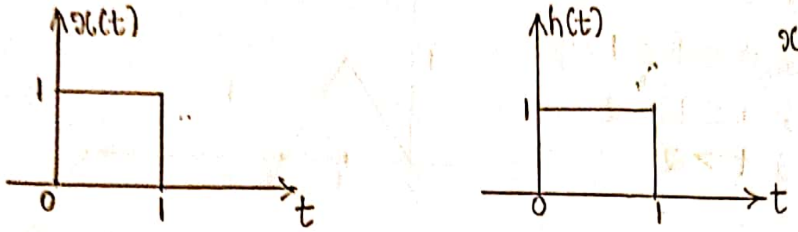
When both the sequences are finite length (CTS):-

A LTI System characterized by impulse response $h(t) = u(t) - u(t-1)$ with an input $x(t) = u(t) - u(t-1)$. Find the Convolution between the two sequences.

(or)

If $x(t) = \begin{cases} 1 ; 0 \le t \le 1 \\ 0 ; \text{otherwise} \end{cases}$ and $h(t) = \begin{cases} 1 ; 0 \le t \le 1 \\ 0 ; \text{otherwise} \end{cases}$

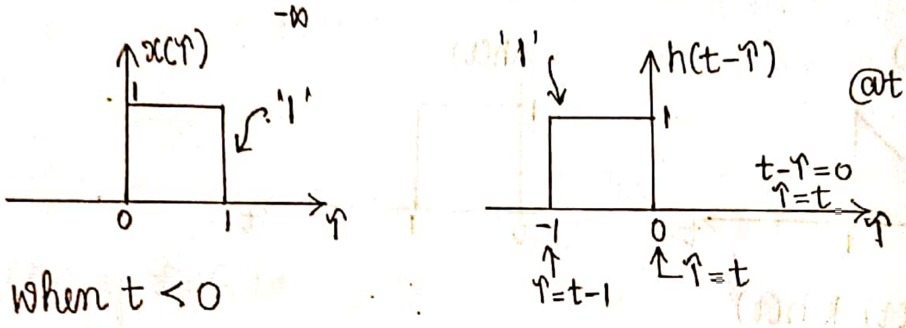
Soln:-



$x(\tau) = 1 ; 0 \le \tau \le 1$
$\quad\quad 0 ; \text{otherwise}$

$h(\tau) = 1 ; 0 \le \tau \le 1$
$\quad\quad 0 ; \text{otherwise}$

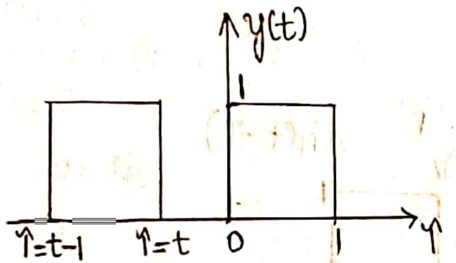
$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$



@t=0

$t - \tau = 0$
$\tau = t$

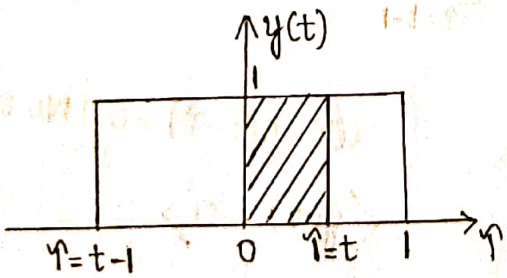
Case (i) :- when $t < 0$



$x(\tau)\, h(t-\tau) = 0$ {No overlap}

$\therefore y(t) = 0 \quad ; t < 0$

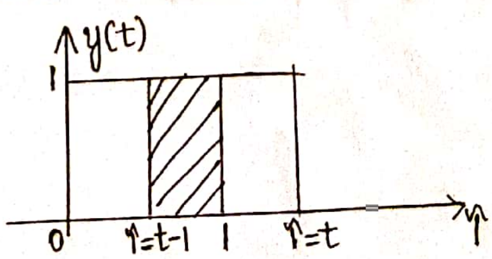
Case (ii) :- when $0 \le t \le 1$



$$y(t) = \int_0^t 1 . d\tau = \tau \Big|_0^t = (t - 0)$$

$$\langle y(t) = t \rangle$$

Case (iii) :- when $1 \le t \le 2$

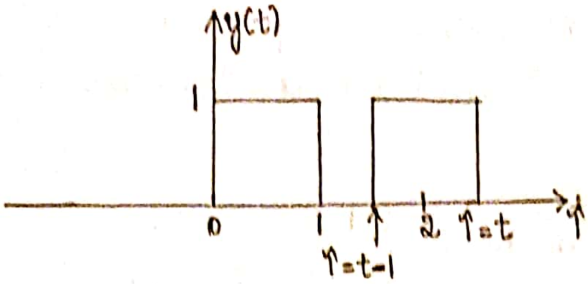


$$y(t) = \int_{t-1}^{1} 1 . d\tau = \tau \Big|_{t-1}^{1}$$

$$= 1 - t + 1$$

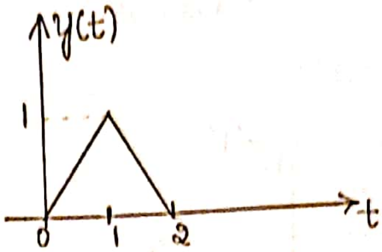
$$\langle y(t) = 2 - t \rangle$$

Case (iv) :- $t-1>1$ (or) $t>2$



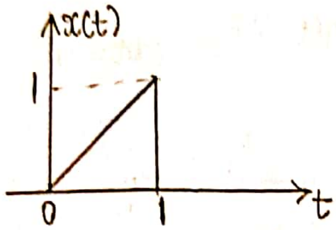
$y(t) = 0$ {No overlap}

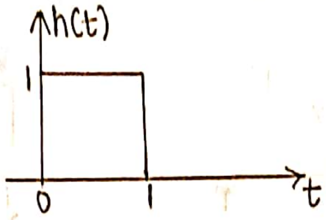
$$\therefore y(t) = \begin{cases} 0 \; ; & t<0 \\ t \; ; & 0 \le t \le 1 \\ 2-t \; ; & 1 \le t \le 2 \\ 0 \; ; & t>2 \end{cases}$$



2) Evaluate the following Continuous-time convolution integral:
$x(t) = t\{u(t)-u(t-1)\}$ and $h(t) = \{u(t)-u(t-1)\}$.

Sol^n:-

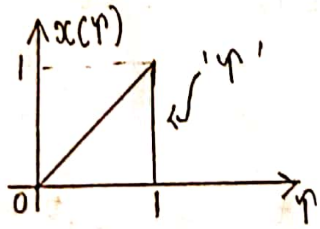


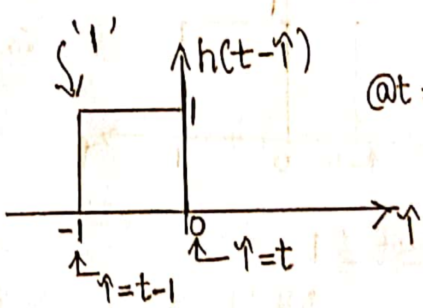


$y(t) = x(t) * h(t)$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

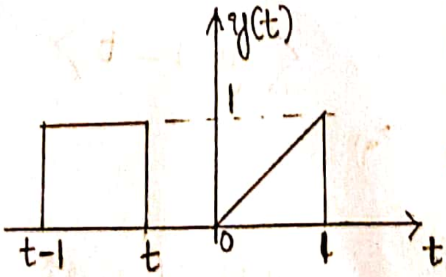
$x(\tau) = \tau$ for $0 \le \tau \le 1$
$\quad 0$; otherwise
$h(\tau) = 1$; $0 \le \tau \le 1$
$\quad 0$; otherwise





@t = 0

Case (i) :- $t<0$



$x(\tau).h(t-\tau) = 0$ {No overlap}

$\langle y(t) = 0 \rangle$

Type - II :-

1) An LTI System characterized by impulse response, $h(t) = \begin{cases} 2 ; & -1 \le t \le 1 \\ 0 ; & \text{else} \end{cases}$ With input $x(t) = \delta(t) + 2\delta(t-1) + \delta(t-2)$. find the Convolution between the two Sequences.

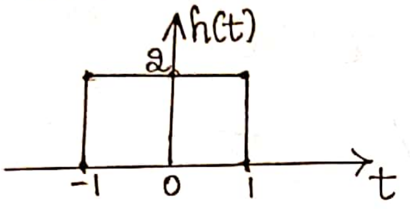
Sol^n:- $y(t) = x(t) * h(t)$

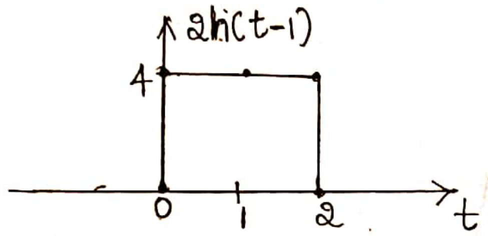
$$= \{\delta(t) + 2\delta(t-1) + \delta(t-2)\} * h(t)$$

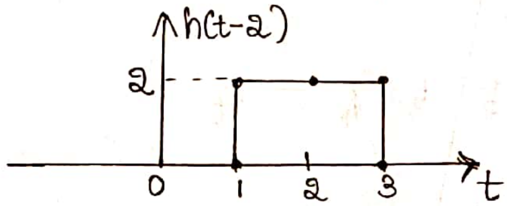
$$y(t) = h(t) + 2h(t-1) + h(t-2).$$

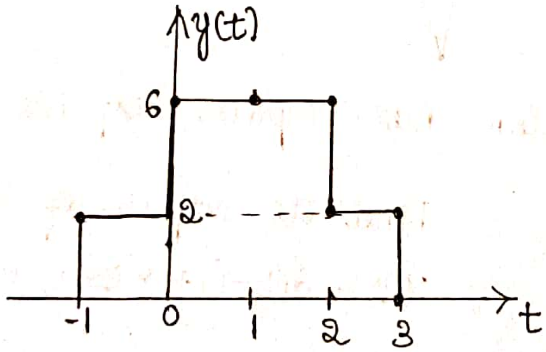
$\because x(t) * \delta(t) = x(t)$

& $x(t) * \delta(t-t_0) = x(t-t_0)$









2) If $x(t) = \delta(t) + 2\delta(t-2) - 10\delta(t-3)$ and $h(t) = \begin{cases} 1+t ; & -1 \le t \le 0 \\ 1-t ; & 0 \le t \le 1. \end{cases}$ find the convolution between the two sequences.

Sol^n:- $y(t) = x(t) * h(t)$

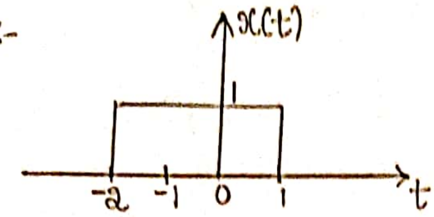
$$= \{\delta(t) + 2\delta(t-2) - 10\delta(t-3)\} * h(t)$$

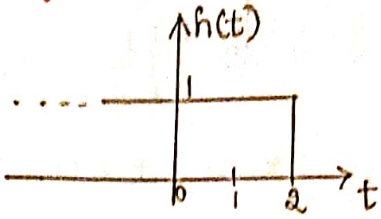
$$y(t) = h(t) + 2h(t-2) - 10h(t-3)$$

When one sequence is finite and another sequence is infinite:-

1) If $x(t) = u(t+2) - u(t-1)$ and $h(t) = u(-t+2)$, find the convolution between the two sequences.

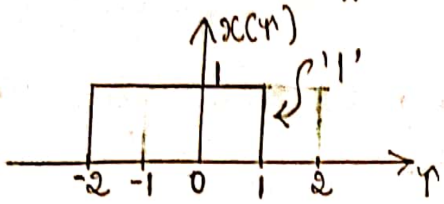
Soln:-

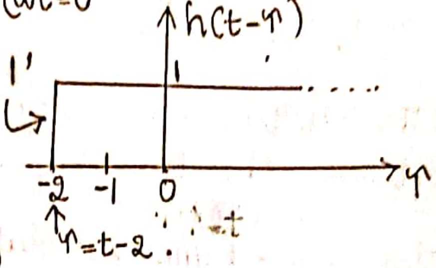




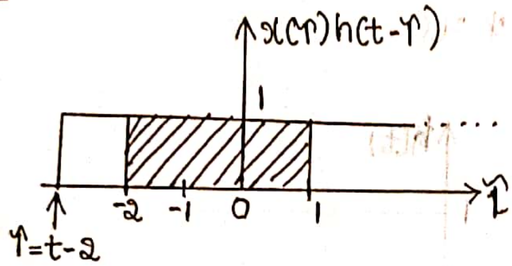
$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau)\, d\tau$$

@t=0





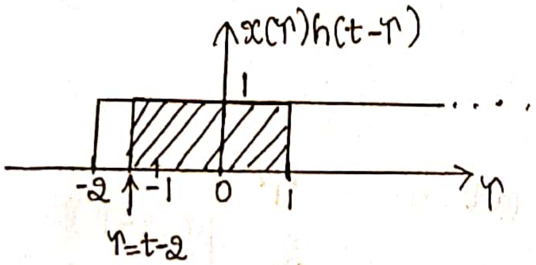
Case (i):- when $t-2 < -2$ (or) $t<0$



$$y(t) = \int_{-2}^{1} 1.1\, d\tau = \tau\Big|_{-2}^{1} = 1-(-2)$$

$$\langle y(t) = 3 \rangle$$

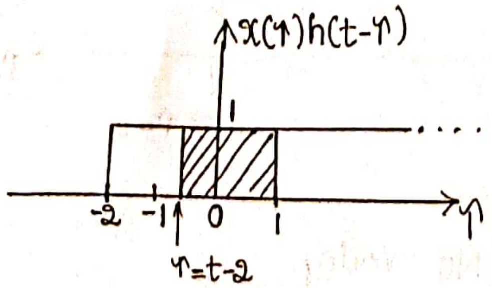
Case (ii):- $-2 \le t-2 \le -1$ (or) $0 \le t \le 1$



$$y(t) = \int_{t-2}^{1} 1.\, d\tau = \tau\Big|_{t-2}^{1} = 1-(t-2)$$

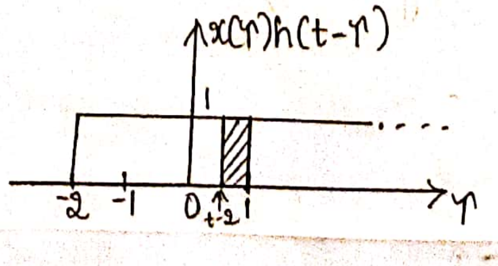
$$\langle y(t) = 3-t \rangle$$

Case (iii):- $-1 \le t-2 \le 0$ (or) $1 \le t \le 2$



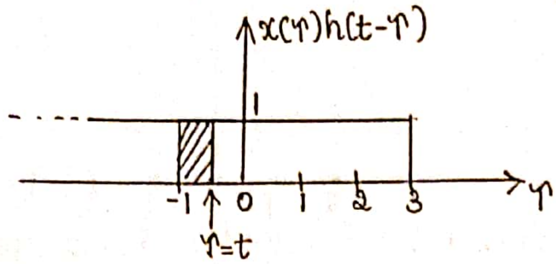
$$y(t) = \int_{t-2}^{1} 1.\, d\tau = \underline{\underline{3-t}}$$

Case (iv):- $0 \le t-2 \le 1$ (or) $2 \le t \le 3$



$$y(t) = \int_{t-2}^{1} 1.\, d\tau = \underline{\underline{3-t}}$$

Case (iii) :- when $-1 \le t \le 0$



$$y(t) = \int_{-1}^{t} \cos(\pi\tau)\cdot 1 \, d\tau$$

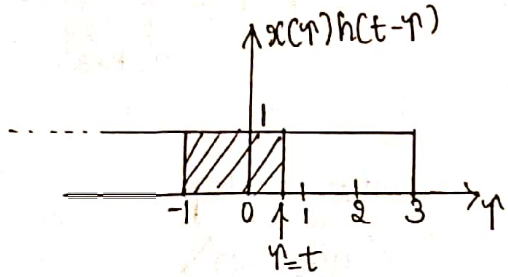
$$= \frac{\sin(\pi\tau)}{\pi}\Bigg|_{-1}^{t}$$

$$= \frac{\sin \pi t}{\pi} - \frac{\sin(-\pi)}{\pi}$$

$$= \frac{\sin \pi t}{\pi} + \cancel{\frac{\sin \pi}{\pi}}^{\,0}$$

$$\left\langle y(t) = \frac{\sin \pi t}{\pi} \right\rangle$$

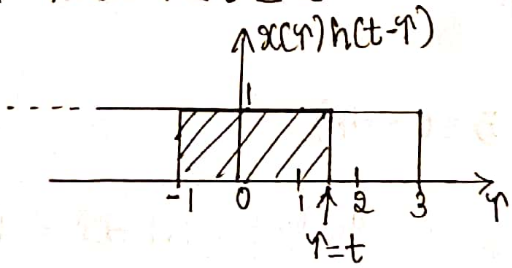
Case (iv) :- when $0 \le t \le 1$



$$y(t) = \int_{-1}^{t} \cos(\pi\tau)\cdot 1 \, d\tau$$

$$\left\langle y(t) = \frac{\sin \pi t}{\pi} \right\rangle$$

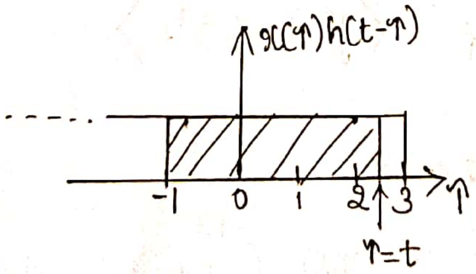
Case (v) :- when $1 \le t \le 2$



$$y(t) = \int_{-1}^{t} \cos(\pi\tau)\cdot 1 \, d\tau$$

$$\left\langle y(t) = \frac{\sin \pi t}{\pi} \right\rangle$$

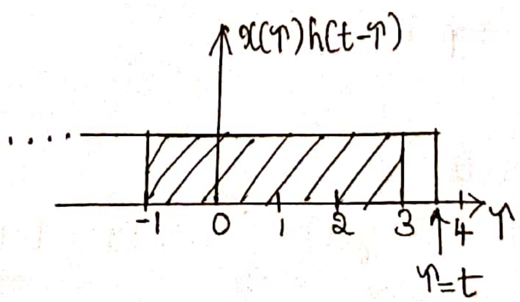
Case (vi) :- when $2 \le t \le 3$



$$y(t) = \int_{-1}^{t} \cos(\pi\tau)\cdot 1 \, d\tau$$

$$\left\langle y(t) = \frac{\sin \pi t}{\pi} \right\rangle$$

Case (vii) :- when $3 \le t \le 4$



$$y(t) = \int_{-1}^{3} \cos(\pi\tau)\cdot 1 \, d\tau$$

$$= \frac{\sin \pi\tau}{\pi}\Bigg|_{-1}^{3} = \frac{1}{\pi}\left[\sin 3\pi + \sin \pi\right]$$

$$\left\langle y(t) = 0 \right\rangle$$